14

Multiple Integration

14.1 Volume and Average Height

Consider a surface $f(x, y)$; you might temporarily think of this as representing physical topography—a hilly landscape, perhaps. What is the average height of the surface (or average altitude of the landscape) over some region?

As with most such problems, we start by thinking about how we might approximate the answer. Suppose the region is a rectangle, $[a, b] \times [c, d]$. We can divide the rectangle into a grid, m subdivisions in one direction and n in the other, as indicated in figure 14.1. We pick x values x_0, x_1,..., x_{m-1} in each subdivision in the x direction, and similarly in the y direction. At each of the points (x_i, y_j) in one of the smaller rectangles in the grid, we compute the height of the surface: $f(x_i, y_j)$. Now the average of these heights should be (depending on the fineness of the grid) close to the average height of the surface:

$$\frac{f(x_0, y_0) + f(x_1, y_0) + \cdots + f(x_0, y_1) + f(x_1, y_1) + \cdots + f(x_{m-1}, y_{n-1})}{mn}.$$

As both m and n go to infinity, we expect this approximation to converge to a fixed value, the actual average height of the surface. For reasonably nice functions this does indeed happen.

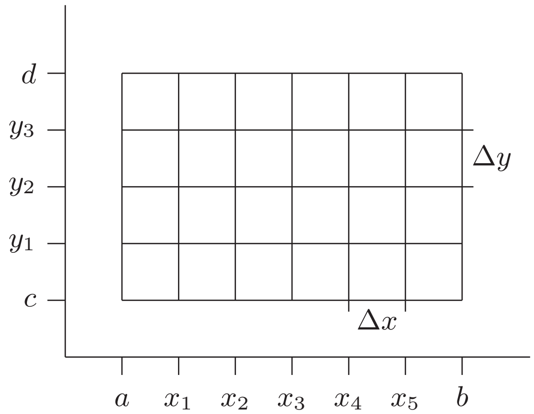


Figure 14.1 A rectangular subdivision of $[a,b]\times[c,d]$.

Using sigma notation, we can rewrite the approximation:

$$\frac{1}{mn}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i) = \frac{1}{(b-a)(d-c)}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i)\frac{b-a}{m}\frac{d-c}{n}$$
$$= \frac{1}{(b-a)(d-c)}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x\Delta y.$$

The two parts of this product have useful meaning: $(b-a)(d-c)$ is of course the area of the rectangle, and the double sum adds up mn terms of the form $f(x_j,y_i)\Delta x\Delta y$, which is the height of the surface at a point times the area of one of the small rectangles into which we have divided the large rectangle. In short, each term $f(x_j,y_i)\Delta x\Delta y$ is the volume of a tall, thin, rectangular box, and is approximately the volume under the surface and above one of the small rectangles; see figure 14.2. When we add all of these up, we get an approximation to the volume under the surface and above the rectangle $R=[a,b]\times[c,d]$. When we take the limit as m and n go to infinity, the double sum becomes the actual volume under the surface, which we divide by $(b-a)(d-c)$ to get the average height.

Double sums like this come up in many applications, so in a way it is the most important part of this example; dividing by $(b-a)(d-c)$ is a simple extra step that allows the computation of an average. As we did in the single variable case, we introduce a special notation for the limit of such a double sum:

$$\lim_{m,n\to\infty}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x\Delta y = \iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA,$$

the **double integral** of f over the region R. The notation dA indicates a small bit of area, without specifying any particular order for the variables x and y; it is shorter and

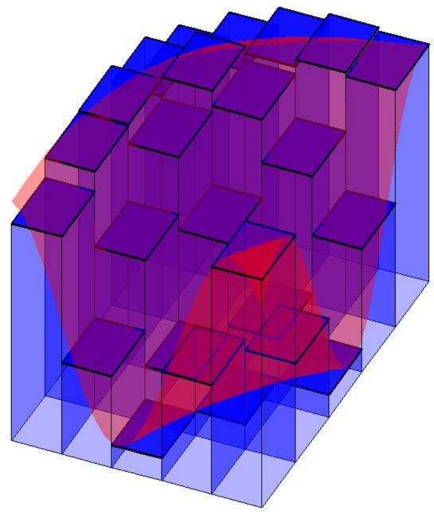


Figure 14.2 Approximating the volume under a surface. (JA)

more "generic" than writing $dx\,dy$. The average height of the surface in this notation is

$$\frac{1}{(b-a)(d-c)}\iint\limits_R f(x,y)\,dA.$$

The next question, of course, is: How do we compute these double integrals? You might think that we will need some two-dimensional version of the Fundamental Theorem of Calculus, but as it turns out we can get away with just the single variable version, applied twice.

Going back to the double sum, we can rewrite it to emphasize a particular order in which we want to add the terms:

$$\sum_{i=0}^{n-1}\left(\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x\right)\Delta y.$$

In the sum in parentheses, only the value of x_j is changing; y_i is temporarily constant. As m goes to infinity, this sum has the right form to turn into an integral:

$$\lim_{m\to\infty}\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x = \int_a^b f(x,y_i)\,dx.$$

So after we take the limit as m goes to infinity, the sum is

$$\sum_{i=0}^{n-1}\left(\int_a^b f(x,y_i)\,dx\right)\Delta y.$$

Of course, for different values of y_i this integral has different values; in other words, it is really a function applied to y_i:

$$G(y)=\int_a^b f(x,y)\,dx.$$

If we substitute back into the sum we get

$$\sum_{i=0}^{n-1} G(y_i)\Delta y.$$

This sum has a nice interpretation. The value $G(y_i)$ is the area of a cross section of the region under the surface $f(x,y)$, namely, when $y=y_i$. The quantity $G(y_i)\Delta y$ can be interpreted as the volume of a solid with face area $G(y_i)$ and thickness Δy. Think of the surface $f(x,y)$ as the top of a loaf of sliced bread. Each slice has a cross-sectional area and a thickness; $G(y_i)\Delta y$ corresponds to the volume of a single slice of bread. Adding these up approximates the total volume of the loaf. (This is very similar to the technique we used to compute volumes in section 9.3, except that there we need the cross-sections to be in some way "the same".) Figure 14.3 shows this "sliced loaf" approximation using the same surface as shown in figure 14.2. Nicely enough, this sum looks just like the sort of sum that turns into an integral, namely,

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=0}^{n-1} G(y_i)\Delta y &= \int_c^d G(y)\,dy\\ &= \int_c^d\int_a^b f(x,y)\,dx\,dy.\end{aligned}$$

Let's be clear about what this means: we first will compute the inner integral, temporarily treating y as a constant. We will do this by finding an anti-derivative with respect to x, then substituting $x=a$ and $x=b$ and subtracting, as usual. The result will be an expression with no x variable but some occurrences of y. Then the outer integral will be an ordinary one-variable problem, with y as the variable.

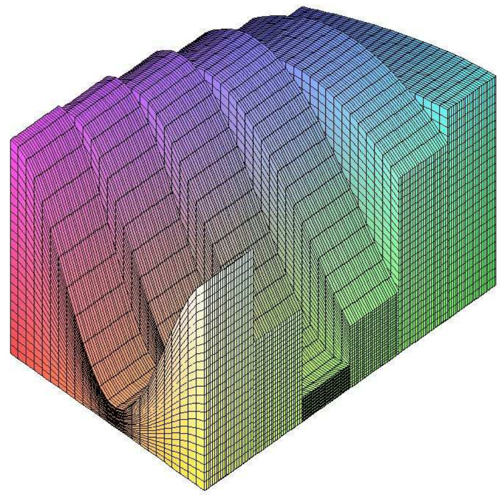


Figure 14.3 Approximating the volume under a surface with slices. (JA)

EXAMPLE 14.1 Figure 14.2 shows the function $\sin(xy) + 6/5$ on $[0.5, 3.5] \times [0.5, 2.5]$. The volume under this surface is

$$\int_{0.5}^{2.5} \int_{0.5}^{3.5} \sin(xy) + \frac{6}{5}\, dx\, dy.$$

The inner integral is

$$\int_{0.5}^{3.5} \sin(xy) + \frac{6}{5}\, dx = \frac{-\cos(xy)}{y} + \frac{6x}{5}\bigg|_{0.5}^{3.5} = \frac{-\cos(3.5y)}{y} + \frac{\cos(0.5y)}{y} + \frac{18}{5}.$$

Unfortunately, this gives a function for which we can't find a simple anti-derivative. To complete the problem we could use Maple or similar software to approximate the integral. Doing this gives a volume of approximately 8.84, so the average height is approximately $8.84/6 \approx 1.47$. □

Because addition and multiplication are commutative and associative, we can rewrite the original double sum:

$$\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(x_j, y_i)\Delta x \Delta y = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} f(x_j, y_i)\Delta y \Delta x.$$

Now if we repeat the development above, the inner sum turns into an integral:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_j,y_i)\Delta y = \int_c^d f(x_j,y)\,dy,$$

and then the outer sum turns into an integral:

$$\lim_{m\to\infty}\sum_{j=0}^{m-1}\left(\int_c^d f(x_j,y)\,dy\right)\Delta x = \int_a^b\int_c^d f(x,y)\,dy\,dx.$$

In other words, we can compute the integrals in either order, first with respect to x then y, or vice versa. Thinking of the loaf of bread, this corresponds to slicing the loaf in a direction perpendicular to the first.

We haven't really proved that the value of a double integral is equal to the value of the corresponding two single integrals in either order of integration, but provided the function is reasonably nice, this is a true theorem, called **Fubini's Theorem**.

EXAMPLE 14.2 We compute $\displaystyle\iint\limits_R 1+(x-1)^2+4y^2\,dA$, where $R=[0,3]\times[0,2]$, in two ways.

First,

$$\begin{aligned}\int_0^3\int_0^2 1+(x-1)^2+4y^2\,dy\,dx &= \int_0^3 y+(x-1)^2y+\frac{4}{3}y^3\Big|_0^2\,dx\\ &= \int_0^3 2+2(x-1)^2+\frac{32}{3}\,dx\\ &= 2x+\frac{2}{3}(x-1)^3+\frac{32}{3}x\Big|_0^3\\ &= 6+\frac{2}{3}\cdot 8+\frac{32}{3}\cdot 3-(0-1\cdot\frac{2}{3}+0)\\ &= 44.\end{aligned}$$

In the other order:

$$\begin{aligned}\int_0^2\int_0^3 1+(x-1)^2+4y^2\,dx\,dy &= \int_0^2 x+\frac{(x-1)^3}{3}+4y^2x\Big|_0^3\,dy\\ &= \int_0^2 3+\frac{8}{3}+12y^2+\frac{1}{3}\,dy\\ &= 3y+\frac{8}{3}y+4y^3+\frac{1}{3}y\Big|_0^2\\ &= 6+\frac{16}{3}+32+\frac{2}{3}\\ &= 44.\end{aligned}$$

□

In this example there is no particular reason to favor one direction over the other; in some cases, one direction might be much easier than the other, so it's usually worth considering the two different possibilities.

Frequently we will be interested in a region that is not simply a rectangle. Let's compute the volume under the surface $x+2y^2$ above the region described by $0\le x\le 1$ and $0\le y\le x^2$, shown in figure 14.4.

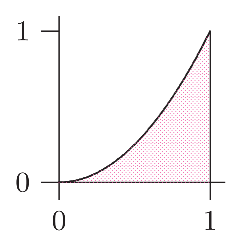


Figure 14.4 A parabolic region of integration.

In principle there is nothing more difficult about this problem. If we imagine the three-dimensional region under the surface and above the parabolic region as an oddly shaped loaf of bread, we can still slice it up, approximate the volume of each slice, and add these volumes up. For example, if we slice perpendicular to the x axis at x_i, the thickness of a slice will be Δx and the area of the slice will be

$$\int_0^{x_i^2} x_i+2y^2\,dy.$$

When we add these up and take the limit as Δx goes to 0, we get the double integral

$$\begin{aligned}\int_0^1\int_0^{x^2} x+2y^2\,dy\,dx &= \int_0^1 xy+\frac{2}{3}y^3\Big|_0^{x^2}\,dx\\ &= \int_0^1 x^3+\frac{2}{3}x^6\,dx\\ &= \frac{x^4}{4}+\frac{2}{21}x^7\Big|_0^1\\ &= \frac{1}{4}+\frac{2}{21}=\frac{29}{84}.\end{aligned}$$

We could just as well slice the solid perpendicular to the y axis, in which case we get

$$\begin{aligned}\int_0^1\int_{\sqrt{y}}^1 x+2y^2\,dx\,dy &= \int_0^1 \frac{x^2}{2}+2y^2x\Big|_{\sqrt{y}}^1\,dy\\ &= \int_0^1 \frac{1}{2}+2y^2-\frac{y}{2}-2y^2\sqrt{y}\,dy\\ &= \frac{y}{2}+\frac{2}{3}y^3-\frac{y^2}{4}-\frac{4}{7}y^{7/2}\Big|_0^1\\ &= \frac{1}{2}+\frac{2}{3}-\frac{1}{4}-\frac{4}{7}=\frac{29}{84}.\end{aligned}$$

What is the average height of the surface over this region? As before, it is the volume divided by the area of the base, but now we need to use integration to compute the area of the base, since it is not a simple rectangle. The area is

$$\int_0^1 x^2\,dx=\frac{1}{3},$$

so the average height is 29/28.

EXAMPLE 14.3 Find the volume under the surface $z=\sqrt{1-x^2}$ and above the triangle formed by $y=x$, $x=1$, and the x-axis.

Let's consider the two possible ways to set this up:

$$\int_0^1\int_0^x \sqrt{1-x^2}\,dy\,dx \qquad \text{or} \qquad \int_0^1\int_y^1 \sqrt{1-x^2}\,dx\,dy.$$

Which appears easier? In the first, the first (inner) integral is easy, because we need an anti-derivative with respect to y, and the entire integrand $\sqrt{1-x^2}$ is constant with respect to y. Of course, the second integral may be more difficult. In the second, the first integral is mildly unpleasant—a trig substitution. So let's try the first one, since the first step is easy, and see where that leaves us.

$$\int_0^1 \int_0^x \sqrt{1-x^2}\,dy\,dx = \int_0^1 y\sqrt{1-x^2}\Big|_0^x\,dx = \int_0^1 x\sqrt{1-x^2}\,dx.$$

This is quite easy, since the substitution $u = 1-x^2$ works:

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{3/2} = -\frac{1}{3}(1-x^2)^{3/2}.$$

Then

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{3/2}\Big|_0^1 = \frac{1}{3}.$$

This is a good example of how the order of integration can affect the complexity of the problem. In this case it is possible to do the other order, but it is a bit messier. In some cases one order may lead to a very difficult or impossible integral; it's usually worth considering both possibilities before going very far. □

Exercises

1. Compute $\displaystyle\int_0^2 \int_0^4 1 + x\,dy\,dx.$ ⇒

2. Compute $\displaystyle\int_{-1}^1 \int_0^2 x + y\,dy\,dx.$ ⇒

3. Compute $\displaystyle\int_1^2 \int_0^y xy\,dx\,dy.$ ⇒

4. Compute $\displaystyle\int_0^1 \int_{y^2/2}^{\sqrt{y}} dx\,dy.$ ⇒

5. Compute $\displaystyle\int_1^2 \int_1^x \frac{x^2}{y^2}\,dy\,dx.$ ⇒

6. Compute $\displaystyle\int_0^1 \int_0^{x^2} \frac{y}{e^x}\,dy\,dx.$ ⇒

7. Compute $\displaystyle\int_0^{\sqrt{\pi/2}} \int_0^{x^2} x\cos y\,dy\,dx.$ ⇒

8. Compute $\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} r^2(\cos\theta - r)\,dr\,d\theta.$ ⇒

9. Compute: $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy$. ⇒

10. Compute: $\displaystyle\int_0^1 \int_{y^2}^1 y\sin(x^2)\,dx\,dy$. ⇒

11. Compute: $\displaystyle\int_0^1 \int_{x^2}^1 x\sqrt{1+y^2}\,dy\,dx$ ⇒

12. Compute: $\displaystyle\int_0^1 \int_0^y \frac{2}{\sqrt{1-x^2}}\,dx\,dy$ ⇒

13. Find the volume bounded by $z = x^2 + y^2$ and $z = 4$. ⇒

14. Find the volume in the first octant bounded by $y^2 = 4 - x$ and $y = 2z$. ⇒

15. Find the volume in the first octant bounded by $y^2 = 4x$, $2x + y = 4$, $z = y$, and $y = 0$. ⇒

16. Find the volume in the first octant bounded by $x + y + z = 9$, $2x + 3y = 18$, and $x + 3y = 9$. ⇒

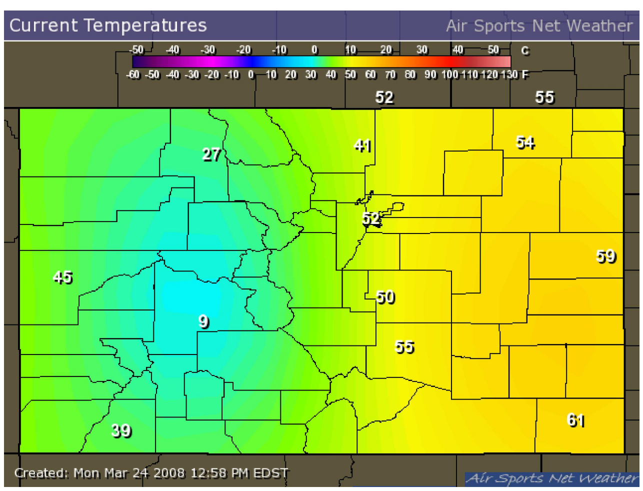
17. Find the volume in the first octant bounded by $x^2 + y^2 = a^2$ and $z = x + y$. ⇒

18. Find the volume bounded by $4x^2 + y^2 = 4z$ and $z = 2$. ⇒

19. Find the volume bounded by $z = x^2 + y^2$ and $z = y$. ⇒

20. Find the average value of $f(x, y) = e^y\sqrt{x + e^y}$ on the rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 1)$ and $(0, 1)$.

21. Below is a weather map of Colorado. Use the data to estimate the average temperature in the state using 4, 16 and 25 subdivisions. Give both an upper and lower estimate. Why do we like Colorado for this problem? What other state might we like?



22. Three cylinders of radius 1 intersect at right angles at the origin, as shown in figure 14.5. Find the volume contained inside all three cylinders. ⇒

23. Prove that if $f(x, y)$ is integrable and if $g(x, y) = \displaystyle\int_a^x \int_b^y f(s, t)\;dt\;ds$ then $g_{xy} = g_{yx} = f(x, y)$.

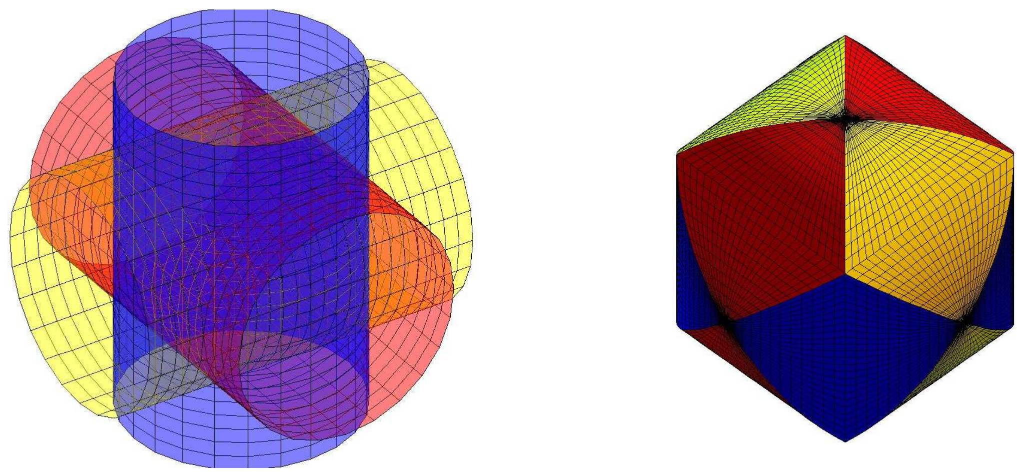


Figure 14.5 Intersection of three cylinders. (JA)

24. Reverse the order of integration on each of the following integrals

a. $\displaystyle\int_0^3 \int_0^{\sqrt{9-y}} f(x,y)\,dx\,dy$

b. $\displaystyle\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx$

c. $\displaystyle\int_0^1 \int_{\arcsin y}^{\pi/2} f(x,y)\,dx\,dy$

25. What are the parallels between Fubini's Theorem and Clairaut's Theorem?

14.2 Double Integrals in Cylindrical Coordinates

Suppose we have a surface given in cylindrical coordinates as $z = f(r,\theta)$ and we wish to find the integral over some region. We could attempt to translate into rectangular coordinates and do the integration there, but it is often easier to stay in cylindrical coordinates.

How might we approximate the volume under such a surface in a way that uses cylindrical coordinates directly? The basic idea is the same as before: we divide the region into many small regions, multiply the area of each small region by the height of the surface somewhere in that little region, and add them up. What changes is the shape of the small regions; in order to have a nice representation in terms of r and θ, we use small pieces of ring-shaped areas, as shown in figure 14.6. Each small region is roughly rectangular, except that two sides are segments of a circle and the other two sides are not quite parallel. Near a point (r,θ), the length of either circular arc is about $r\Delta\theta$ and the length of each straight side is simply Δr. When Δr and $\Delta\theta$ are very small, the region is nearly a rectangle with area $r\Delta r\Delta\theta$, and the volume under the surface is approximately

$$\sum\sum f(r_i,\theta_j)r_i\Delta r\Delta\theta.$$

In the limit, this turns into a double integral

$$\int_{\theta_0}^{\theta_1}\int_{r_0}^{r_1} f(r,\theta)r\,dr\,d\theta.$$

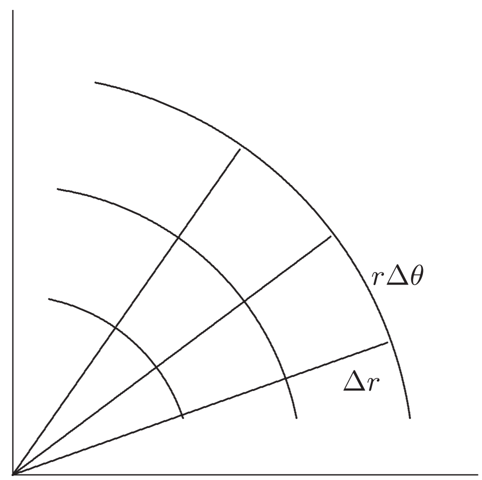


Figure 14.6 A cylindrical coordinates "grid".

EXAMPLE 14.4 Find the volume under $z=\sqrt{4-r^2}$ above the quarter circle bounded by the two axes and the circle $x^2+y^2=4$ in the first quadrant.

In terms of r and θ, this region is described by the restrictions $0\le r\le 2$ and $0\le\theta\le\pi/2$, so we have

$$\begin{aligned}\int_0^{\pi/2}\int_0^2\sqrt{4-r^2}\,r\,dr\,d\theta &= \int_0^{\pi/2} -\frac{1}{3}(4-r^2)^{3/2}\Big|_0^2\,d\theta\\ &= \int_0^{\pi/2}\frac{8}{3}\,d\theta\\ &= \frac{4\pi}{3}.\end{aligned}$$

The surface is a portion of the sphere of radius 2 centered at the origin, in fact exactly one-eighth of the sphere. We know the formula for volume of a sphere is $(4/3)\pi r^3$, so the volume we have computed is $(1/8)(4/3)\pi 2^3=(4/3)\pi$, in agreement with our answer. □

This example is much like a simple one in rectangular coordinates: the region of interest may be described exactly by a constant range for each of the variables. As with rectangular coordinates, we can adapt the method to deal with more complicated regions.

EXAMPLE 14.5 Find the volume under $z = \sqrt{4 - r^2}$ above the region enclosed by the curve $r = 2\cos\theta$, $-\pi/2 \le \theta \le \pi/2$; see figure 14.7. The region is described in polar coordinates by the inequalities $-\pi/2 \le \theta \le \pi/2$ and $0 \le r \le 2\cos\theta$, so the double integral is

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2}\, r\, dr\, d\theta = 2\int_0^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2}\, r\, dr\, d\theta.$$

We can rewrite the integral as shown because of the symmetry of the volume; this avoids a complication during the evaluation. Proceeding:

$$\begin{aligned} 2\int_0^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2}\, r\, dr\, d\theta &= 2\int_0^{\pi/2} -\frac{1}{3}\,(4 - r^2)^{3/2}\Big|_0^{2\cos\theta}\, d\theta \\ &= 2\int_0^{\pi/2} -\frac{8}{3}\sin^3\theta + \frac{8}{3}\, d\theta \\ &= 2\left(-\frac{8}{3}\frac{\cos^3\theta}{3} - \cos\theta + \frac{8}{3}\theta\Big|_0^{\pi/2}\right. \\ &= \frac{8}{3}\pi - \frac{32}{9}. \end{aligned}$$

□

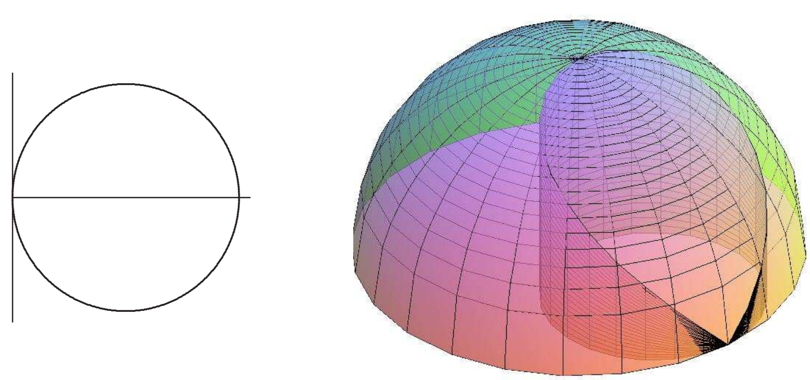


Figure 14.7 Volume over a region with non-constant limits.

You might have learned a formula for computing areas in polar coordinates. It is possible to compute areas as volumes, so that you need only remember one technique. Consider the surface $z = 1$, a horizontal plane. The volume under this surface and above a region in the x-y plane is simply $1 \cdot (\text{area of the region})$, so computing the volume really just computes the area of the region.

EXAMPLE 14.6 Find the area outside the circle $r = 2$ and inside $r = 4\sin\theta$; see figure 14.8. The region is described by $\pi/6 \le \theta \le 5\pi/6$ and $2 \le r \le 4\sin\theta$, so the integral is

$$\begin{aligned}\int_{\pi/6}^{5\pi/6}\int_{2}^{4\sin\theta} 1\, r\, dr\, d\theta &= \int_{\pi/6}^{5\pi/6} \left.\frac{1}{2}r^2\right|_{2}^{4\sin\theta} d\theta \\ &= \int_{\pi/6}^{5\pi/6} 8\sin^2\theta - 2\, d\theta \\ &= \frac{4}{3}\pi + 2\sqrt{3}.\end{aligned}$$

□

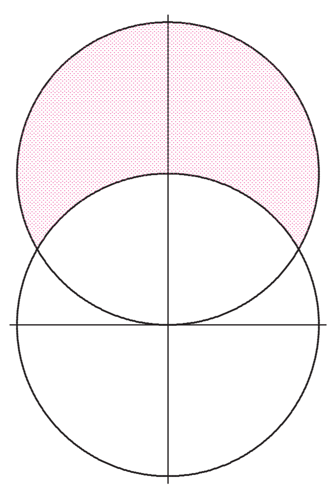


Figure 14.8 Finding area by computing volume.

Exercises

1. Find the volume above the x-y plane, under the surface $r^2 = 2z$, and inside $r = 2$. ⇒

2. Find the volume inside both $r = 1$ and $r^2 + z^2 = 4$. ⇒

3. Find the volume below $z = \sqrt{1 - r^2}$ and above the top half of the cone $z = r$. ⇒

4. Find the volume below $z = r$, above the x-y plane, and inside $r = \cos\theta$. ⇒

5. Find the volume below $z = r$, above the x-y plane, and inside $r = 1 + \cos\theta$. ⇒

6. Find the volume between $x^2 + y^2 = z^2$ and $x^2 + y^2 = z$. ⇒

7. Find the area inside $r = 1 + \sin\theta$ and outside $r = 2\sin\theta$. ⇒

8. Find the area inside both $r = 2\sin\theta$ and $r = 2\cos\theta$. ⇒

9. Find the area inside the four-leaf rose $r = \cos(2\theta)$ and outside $r = 1/2$. ⇒

10. Investigate and describe the differences between the graphs of $r = \cos(2\theta)$ and $r = \sin(2\theta)$.

11. Investigate and describe the differences between the graphs of $r = \cos(2k\theta)$ and $r = \cos((2k+1)\theta)$

12. Figure 14.9 shows the plot of $r = 1 + 4\sin(5\theta)$.

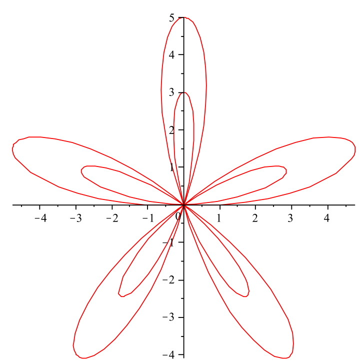


Figure 14.9 $r = 1 + 4\sin(5\theta)$

a. Describe the behavior of the graph in terms of the given equation. Specifically, explain maximum and minimum values, number of leaves, and the 'leaves within leaves'.

b. Give an integral or integrals to determine the area outside a smaller leaf but inside a larger leaf.

c. How would changing the value of a in the equation $r = 1 + a\cos(5\theta)$ change the relative sizes of the inner and outer leaves? Focus on values $a \geq 1$. (Hint: How would we change the maximum and minimum values?)

13. Consider the integral $\displaystyle\iint_D \frac{1}{\sqrt{x^2 + y^2}}\, dA$, where D is the unit disk centered at the origin.

a. Why might this integral be considered improper?

Exercises

1. Find the area of the surface of a right circular cone of height h and base radius a. ⇒

2. Find the area of the portion of the plane $z = mx$ inside the cylinder $x^2 + y^2 = a^2$. ⇒

3. Find the area of the portion of the plane $x + y + z = 1$ in the first octant. ⇒

4. Find the area of the upper half of the cone $x^2 + y^2 = z^2$ inside the cylinder $x^2 + y^2 - 2x = 0$. ⇒

5. Find the area of the upper half of the cone $x^2 + y^2 = z^2$ above the interior of one loop of $r = \cos(2\theta)$. ⇒

6. Find the area of the upper hemisphere of $x^2 + y^2 + z^2 = 1$ above the interior of one loop of $r = \cos(2\theta)$. ⇒

7. The plane $ax + by + cz = d$ cuts a triangle in the first octant provided that a, b, c and d are all positive. Set up the integral to find the area of this triangle.

8. The surface area formula can be used to compute the surface area of the upper half of the sphere $x^2 + y^2 + z^2 = a^2$, but the integral is improper.

- **a.** Set up the appropriate integral to calculate this area and give two algebraic reasons why it is improper.
- **b.** Find the surface area of the upper hemisphere of $x^2 + y^2 + z^2 = a^2$ above a circle of radius t where $t < a$.
- **c.** Find the surface area of the whole upper hemisphere by taking a limit of your answer in part (b) as t approaches a.

14.5 Triple Integrals

It will come as no surprise that we can also do triple integrals—integrals over a three-dimensional region. The simplest application allows us to compute volumes in an alternate way.

To approximate a volume in three dimensions, we can divide the three-dimensional region into small rectangular boxes, each $\Delta x \times \Delta y \times \Delta z$ with volume $\Delta x \Delta y \Delta z$. Then we add them all up and take the limit, to get an integral:

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} dz\, dy\, dx.$$

Of course, if the limits are constant, we are simply computing the volume of a rectangular box.

EXAMPLE 14.10 We use an integral to compute the volume of the box with opposite corners at $(0, 0, 0)$ and $(1, 2, 3)$.

$$\int_0^1 \int_0^2 \int_0^3 dz\, dy\, dx = \int_0^1 \int_0^2 z|_0^3\, dy\, dx = \int_0^1 \int_0^2 3\, dy\, dx = \int_0^1 3y|_0^2\, dx = \int_0^1 6\, dx = 6.$$

□

Of course, this is more interesting and useful when the limits are not constant.

EXAMPLE 14.11 Find the volume of the tetrahedron with corners at $(0,0,0)$, $(0,3,0)$, $(2,3,0)$, and $(2,3,5)$.

The whole problem comes down to correctly describing the region by inequalities: $0 \le x \le 2$, $3x/2 \le y \le 3$, $0 \le z \le 5x/2$. The lower y limit comes from the equation of the line $y = 3x/2$ that forms one edge of the tetrahedron in the x-y plane; the upper z limit comes from the equation of the plane $z = 5x/2$ that forms the "upper" side of the tetrahedron. Now the volume is

$$\begin{aligned}\int_0^2 \int_{3x/2}^3 \int_0^{5x/2} dz\,dy\,dx &= \int_0^2 \int_{3x/2}^3 z\Big|_0^{5x/2}\,dy\,dx \\ &= \int_0^2 \int_{3x/2}^3 \frac{5x}{2}\,dy\,dx \\ &= \int_0^2 \frac{5x}{2}y\Big|_{3x/2}^3\,dx \\ &= \int_0^2 \frac{15x}{2} - \frac{15x^2}{4}\,dx \\ &= \frac{15x^2}{4} - \frac{15x^3}{12}\Big|_0^2 \\ &= 15 - 10 = 5.\end{aligned}$$

□

Pretty much just the way we did for two dimensions we can use triple integration to compute mass, center of mass, and various average quantities.

EXAMPLE 14.12 Suppose the temperature at a point is given by $T = xyz$. Find the average temperature in the cube with opposite corners at $(0,0,0)$ and $(2,2,2)$.

In two dimensions we add up the temperature at "each" point and divide by the area; here we add up the temperatures and divide by the volume, 8:

$$\begin{aligned}\frac{1}{8}\int_0^2\int_0^2\int_0^2 xyz\,dz\,dy\,dx &= \frac{1}{8}\int_0^2\int_0^2 \frac{xyz^2}{2}\Big|_0^2\,dy\,dx = \frac{1}{16}\int_0^2\int_0^2 xy\,dy\,dx \\ &= \frac{1}{4}\int_0^2 \frac{xy^2}{2}\Big|_0^2\,dx = \frac{1}{8}\int_0^2 4x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_0^2 = 1.\end{aligned}$$

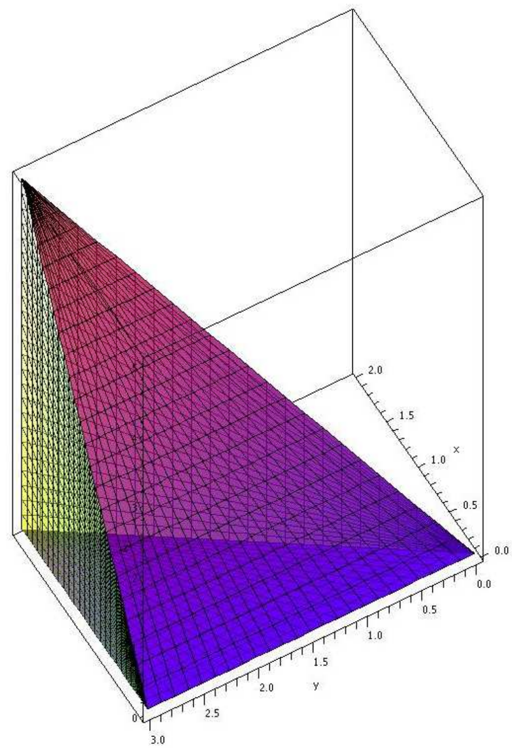


Figure 14.11 A tetrahedron. (JA)

□

EXAMPLE 14.13 Suppose the density of an object is given by xz, and the object occupies the tetrahedron with corners $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(0,1,1)$. Find the mass and center of mass of the object.

As usual, the mass is the integral of density over the region:

$$M = \int_0^1 \int_x^1 \int_0^{y-x} xz\,dz\,dy\,dx = \int_0^1 \int_x^1 \frac{x(y-x)^2}{2}\,dy\,dx = \frac{1}{2}\int_0^1 \frac{x(1-x)^3}{3}\,dx$$
$$= \frac{1}{6}\int_0^1 x - 3x^2 + 3x^3 - x^4\,dx = \frac{1}{120}.$$

We compute moments as before, except now there is a third moment:

$$M_{xy} = \int_0^1 \int_x^1 \int_0^{y-x} xz^2 \, dz \, dy \, dx = \frac{1}{360},$$
$$M_{xz} = \int_0^1 \int_x^1 \int_0^{y-x} xyz \, dz \, dy \, dx = \frac{1}{144},$$
$$M_{yz} = \int_0^1 \int_x^1 \int_0^{y-x} x^2 z \, dz \, dy \, dx = \frac{1}{360}.$$

Finally, the coordinates of the center of mass are $\bar{x} = M_{yz}/M = 1/3$, $\bar{y} = M_{xz}/M = 5/6$, and $\bar{z} = M_{xy}/M = 1/3$. □

Exercises

1. Evaluate $\int_0^1 \int_0^x \int_0^{x+y} 2x + y - 1 \, dz \, dy \, dx$. ⇒

2. Evaluate $\int_0^2 \int_{-1}^{x^2} \int_1^y xyz \, dz \, dy \, dx$. ⇒

3. Evaluate $\int_0^1 \int_0^x \int_0^{\ln y} e^{x+y+z} \, dz \, dy \, dx$. ⇒

4. Evaluate $\int_0^{\pi/2} \int_0^{\sin\theta} \int_0^{r\cos\theta} r^2 \, dz \, dr \, d\theta$. ⇒

5. Evaluate $\int_0^{\pi} \int_0^{\sin\theta} \int_0^{r\sin\theta} r\cos^2\theta \, dz \, dr \, d\theta$. ⇒

6. Evaluate $\int_0^1 \int_0^{y^2} \int_0^{x+y} x \, dz \, dx \, dy$. ⇒

7. Evaluate $\int_1^2 \int_y^{y^2} \int_0^{\ln(y+z)} e^x \, dx \, dz \, dy$. ⇒

8. For each of the integrals in the previous exercises, give a description of the volume (both algebraic and geometric) that is the domain of integration.

9. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner. ⇒

10. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge. ⇒

11. An object occupies the volume of the upper hemisphere of $x^2 + y^2 + z^2 = 4$ and has density z at (x, y, z). Find the center of mass. ⇒

12. An object occupies the volume of the pyramid with corners at $(1, 1, 0)$, $(1, -1, 0)$, $(-1, -1, 0)$, $(-1, 1, 0)$, and $(0, 0, 2)$ and has density $x^2 + y^2$ at (x, y, z). Find the center of mass. ⇒

13. Verify the moments M_{xy}, M_{xz}, and M_{yz} of example 14.13 by evaluating the integrals.

14. Find the region E for which $\iiint\limits_E (1 - x^2 - y^2 - z^2)\, dV$ is a maximum.

14.6 Cylindrical and Spherical Coordinates

We have seen that sometimes double integrals are simplified by doing them in polar coordinates; not surprisingly, triple integrals are sometimes simpler in cylindrical coordinates or spherical coordinates. To set up integrals in polar coordinates, we had to understand the shape and area of a typical small region into which the region of integration was divided. We need to do the same thing here, for three dimensional regions.

The cylindrical coordinate system is the simplest, since it is just the polar coordinate system plus a z coordinate. A typical small unit of volume is the shape shown in figure 14.6 "fattened up" in the z direction, so its volume is $r\Delta r\Delta\theta\Delta z$, or in the limit, $r\,dr\,d\theta\,dz$.

EXAMPLE 14.14 Find the volume under $z = \sqrt{4 - r^2}$ above the quarter circle inside $x^2 + y^2 = 4$ in the first quadrant.

We could of course do this with a double integral, but we'll use a triple integral:

$$\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 \sqrt{4-r^2}\; r\,dr\,d\theta = \frac{4\pi}{3}.$$

Compare this to example 14.4. □

EXAMPLE 14.15 An object occupies the space inside both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$, and has density x^2 at (x, y, z). Find the total mass.

We set this up in cylindrical coordinates, recalling that $x = r\cos\theta$:

$$\begin{aligned}\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3\cos^2(\theta)\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^1 2\sqrt{4-r^2}\; r^3\cos^2(\theta)\,dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{128}{15} - \frac{22}{5}\sqrt{3}\right)\cos^2(\theta)\,d\theta \\ &= \left(\frac{128}{15} - \frac{22}{5}\sqrt{3}\right)\pi\end{aligned}$$

□

Spherical coordinates are somewhat more difficult to understand. The small volume we want will be defined by $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$, as pictured in figure 14.12. To gain a better understanding, see the Java applet. The small volume is nearly box shaped, with 4 flat sides and two sides formed from bits of concentric spheres. When $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$ are all very small, the volume of this little region will be nearly the volume we get by treating it as a box. One dimension of the box is simply $\Delta\rho$, the change in distance from the origin. The other two dimensions are the lengths of small circular arcs, so they are $r\Delta\alpha$ for some suitable r and α, just as in the polar coordinates case.

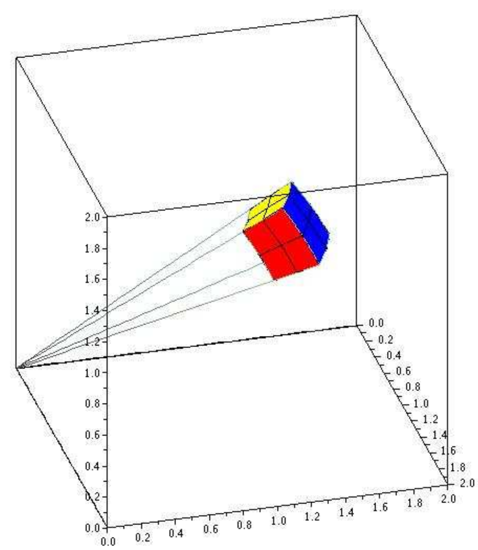


Figure 14.12 A small unit of volume for spherical coordinates. (JA)

The easiest of these to understand is the arc corresponding to a change in ϕ, which is nearly identical to the derivation for polar coordinates, as shown in the left graph in figure 14.13. In that graph we are looking "face on" at the side of the box we are interested in, so the small angle pictured is precisely $\Delta\phi$, the vertical axis really is the z axis, but the horizontal axis is *not* a real axis—it is just some line in the x-y plane. Because the other arc is governed by θ, we need to imagine looking straight down the z axis, so that the apparent angle we see is $\Delta\theta$. In this view, the axes really are the x and y axes. In this graph, the apparent distance from the origin is not ρ but $\rho\sin\phi$, as indicated in the left graph.

The upshot is that the volume of the little box is approximately $\Delta\rho(\rho\Delta\phi)(\rho\sin\phi\Delta\theta) = \rho^2\sin\phi\Delta\rho\Delta\phi\Delta\theta$, or in the limit $\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$.

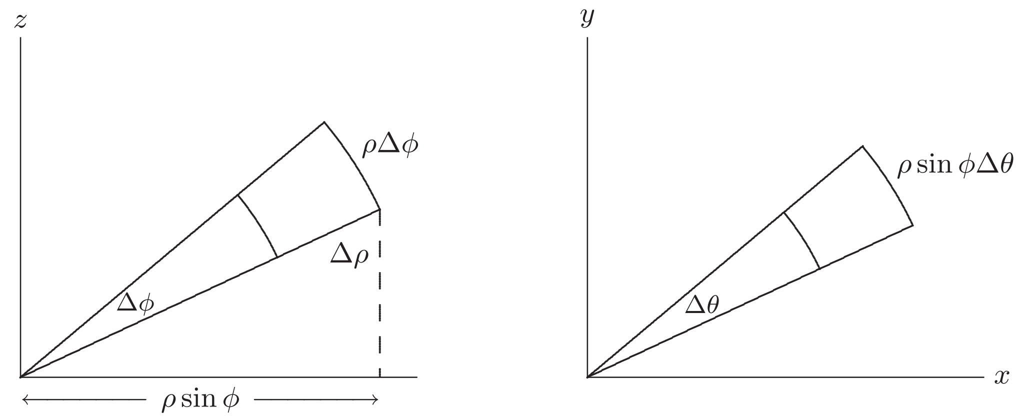


Figure 14.13 Setting up integration in spherical coordinates.

EXAMPLE 14.16 Suppose the temperature at (x, y, z) is $T = 1/(1 + x^2 + y^2 + z^2)$. Find the average temperature in the unit sphere centered at the origin.

In two dimensions we add up the temperature at "each" point and divide by the area; here we add up the temperatures and divide by the volume, $(4/3)\pi$:

$$\frac{3}{4\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2}\,dz\,dy\,dx$$

This looks quite messy; since everything in the problem is closely related to a sphere, we'll convert to spherical coordinates.

$$\frac{3}{4\pi}\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{1} \frac{1}{1+\rho^2}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{3}{4\pi}(4\pi-\pi^2) = 3-\frac{3\pi}{4}.$$

□