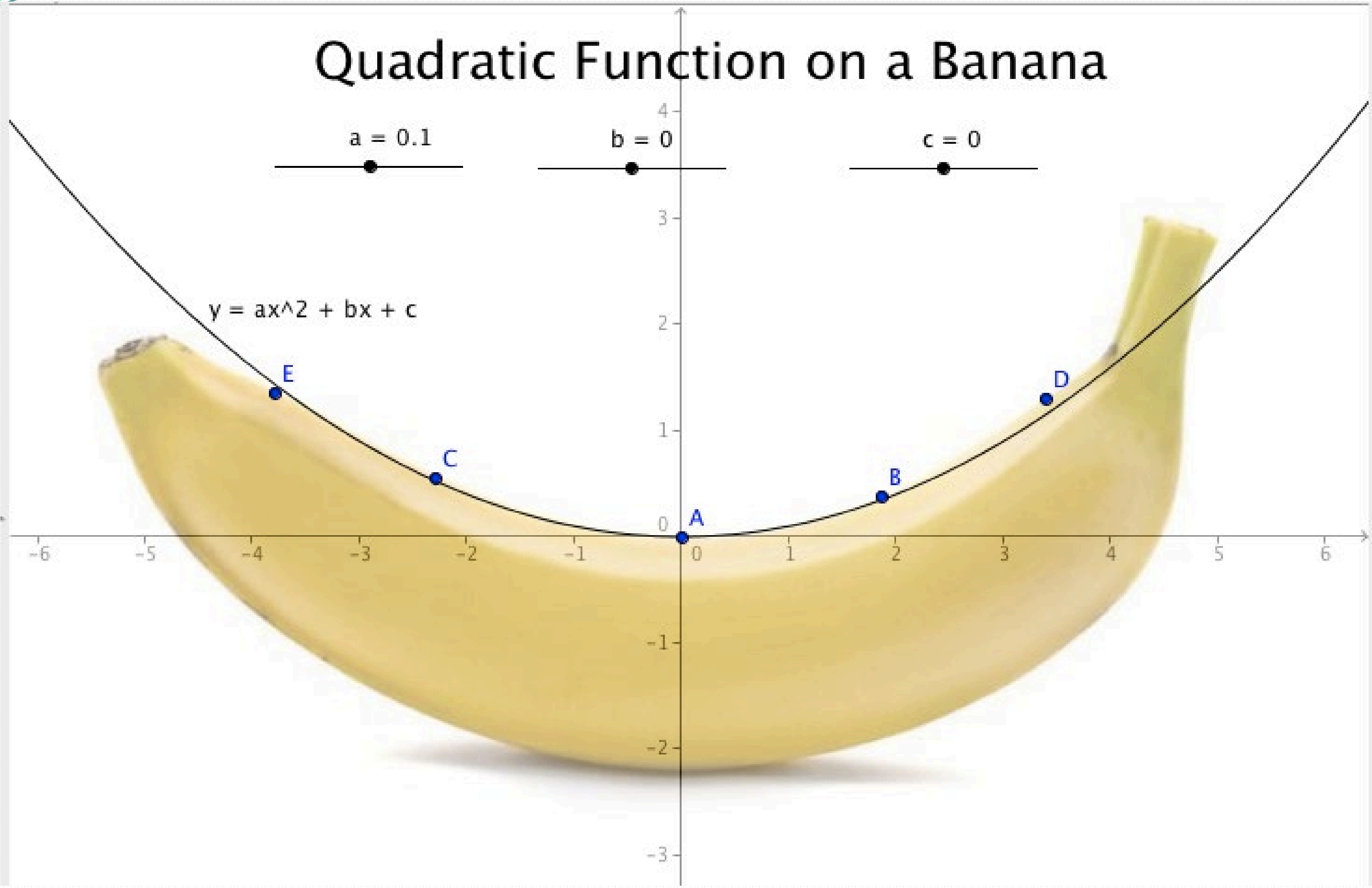
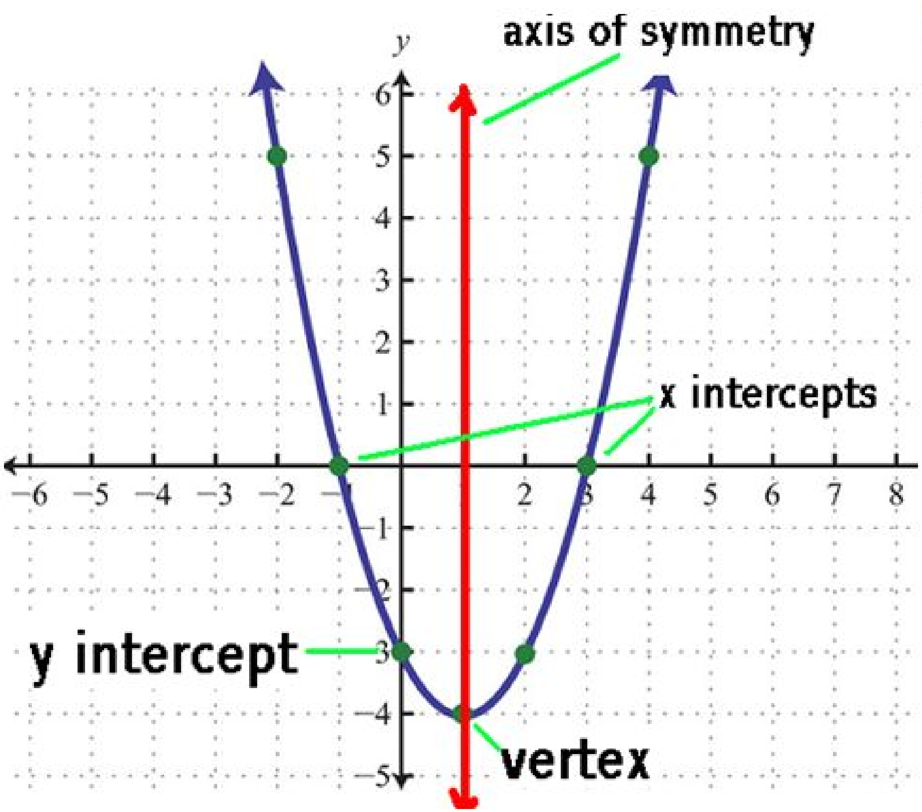
MULTIVARIABLE CALCULUS

QUADRIC LINES


Quadratic Function on a Banana
a = 0.1
b = 0
c = 0
y = ax^2 + bx + c
E
C
A
B
D
4
3
2
1
0
-1
-2
-3
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6

Precalculus Section 1.7

Define and graph quadratic functions



Any function that can be written in the form:

$y = ax^2 + bx + c$ is called a quadratic function. It's graph is called a parabola.

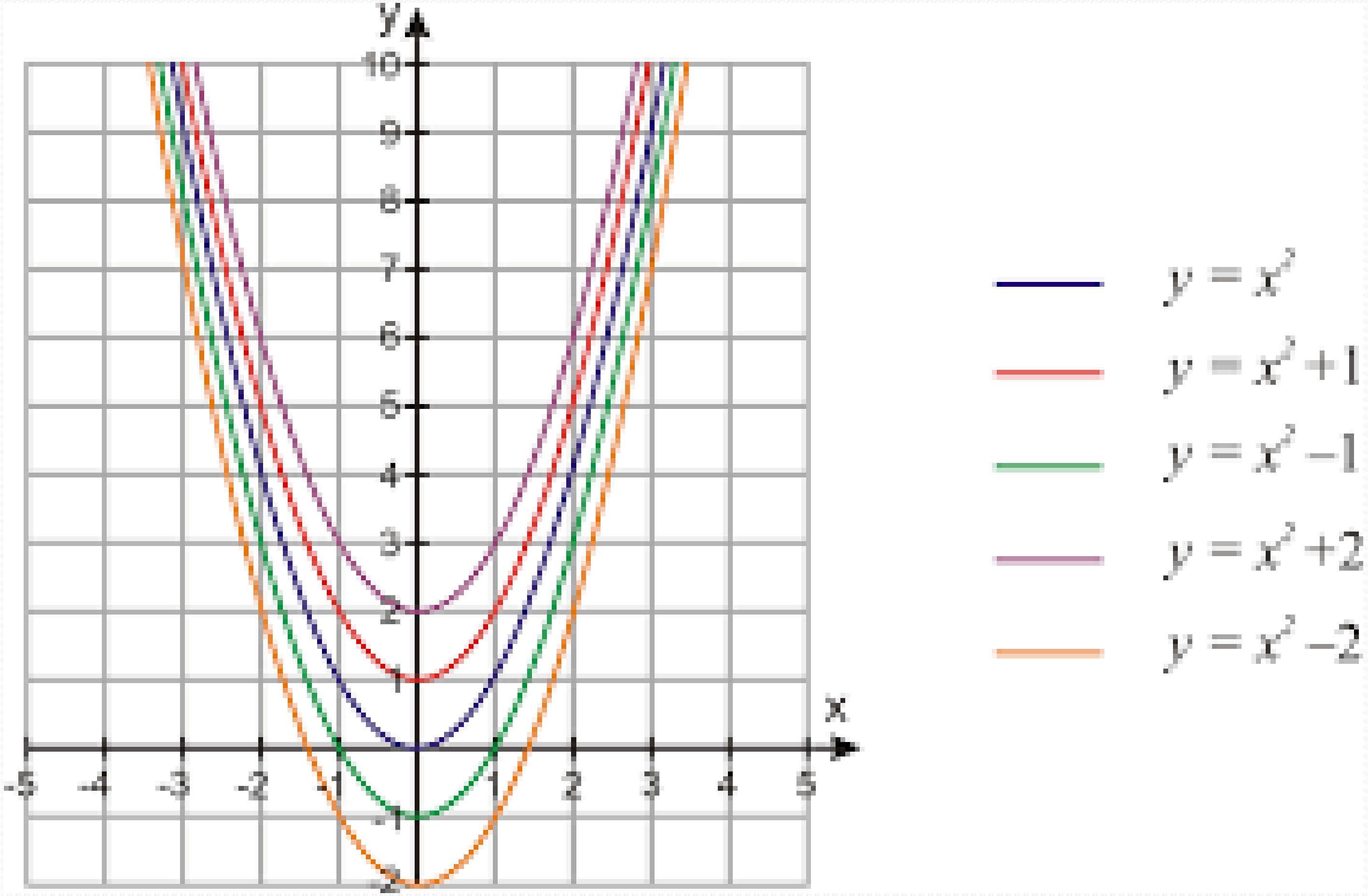
Consider the graphs of the quadratic functions:

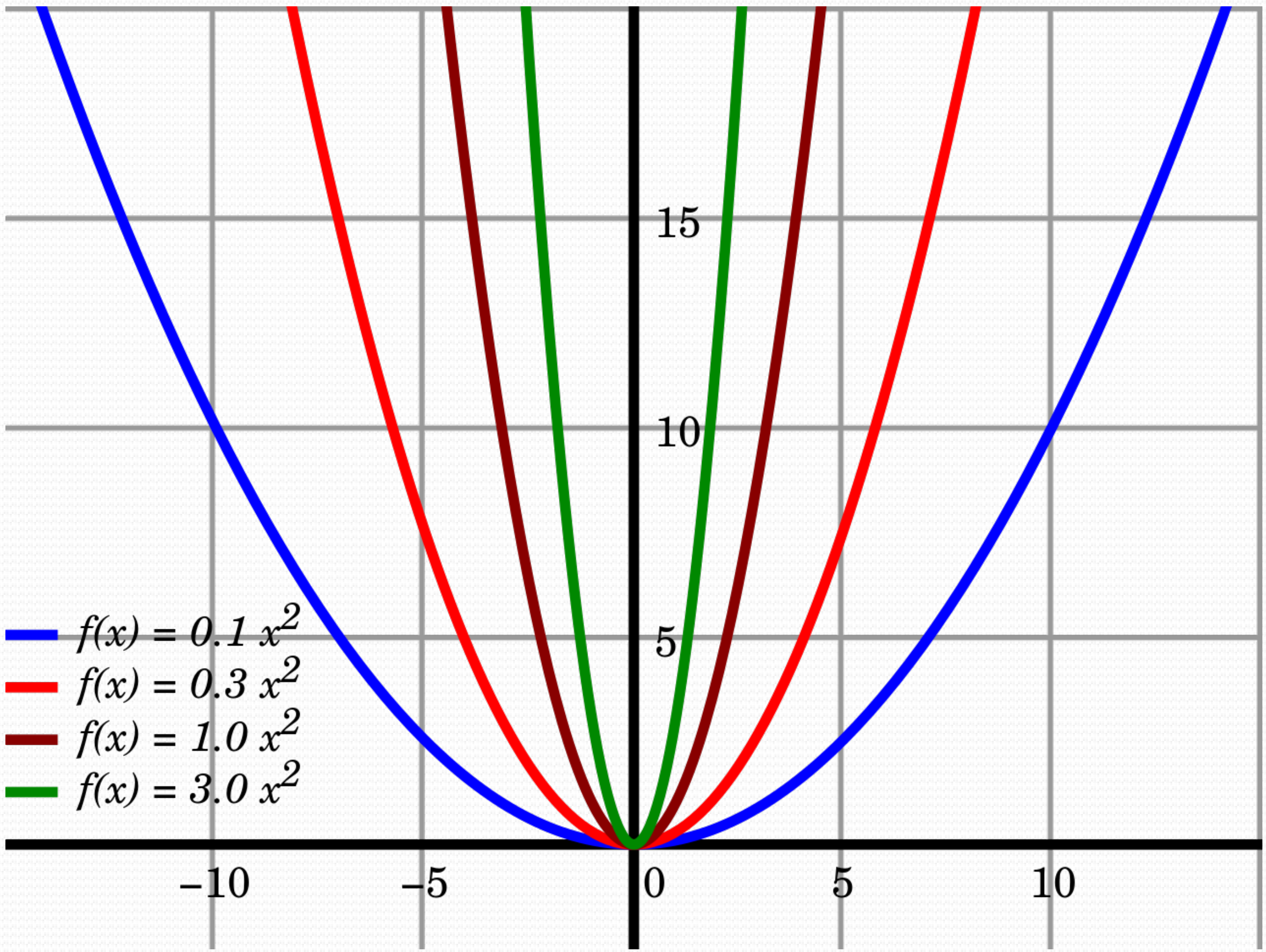
$y = x^2$

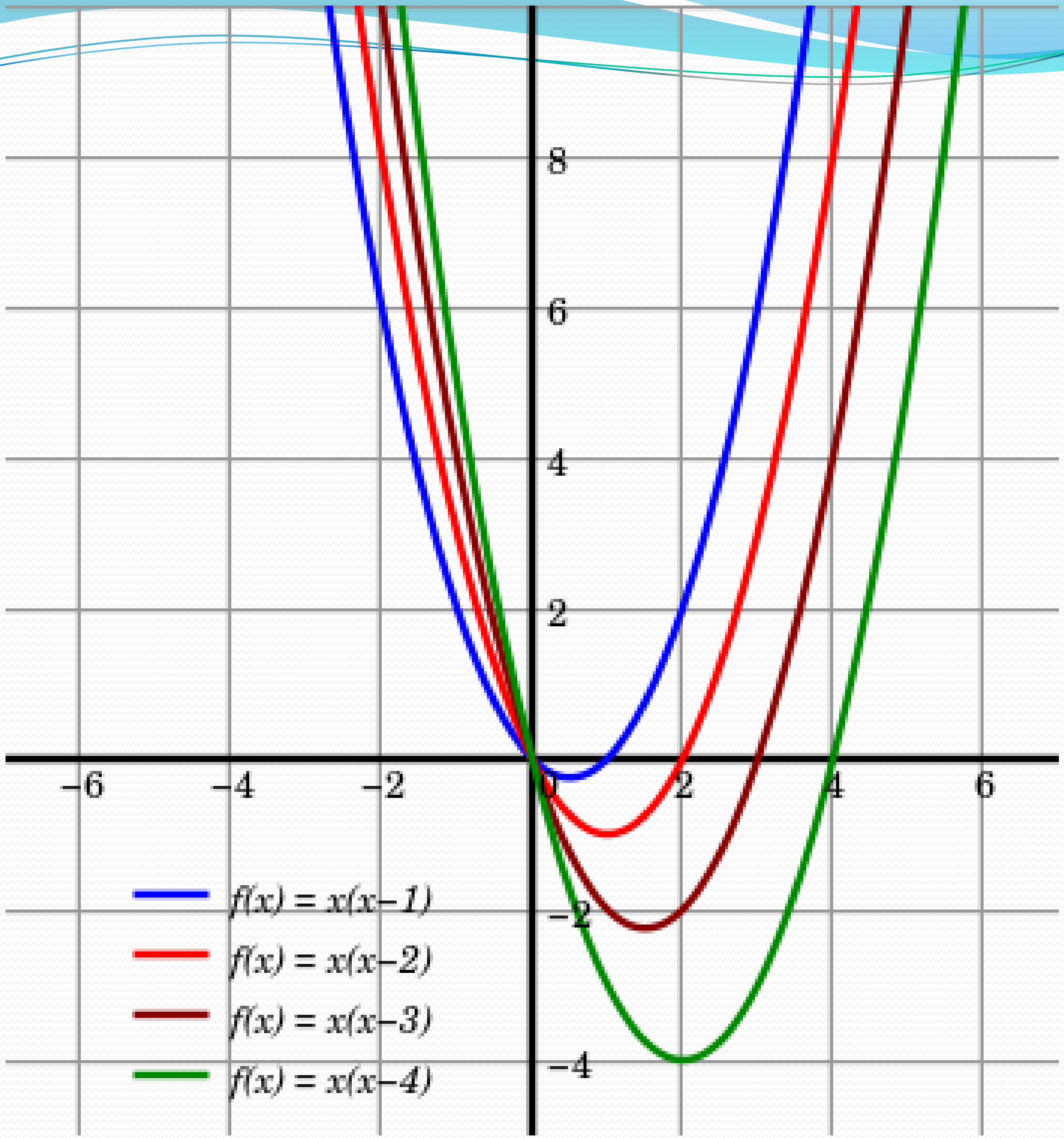
$y = 2x^2$

$y = -2x^2$

$y = x^2 - x - 6$


y
10
9
8
7
6
5
4
3
2
1
-1
-2
x
-5
-4
-3
-2
-1
1
2
3
4
5
$y = x^2$
$y = x^2 + 1$
$y = x^2 - 1$
$y = x^2 + 2$
$y = x^2 - 2$

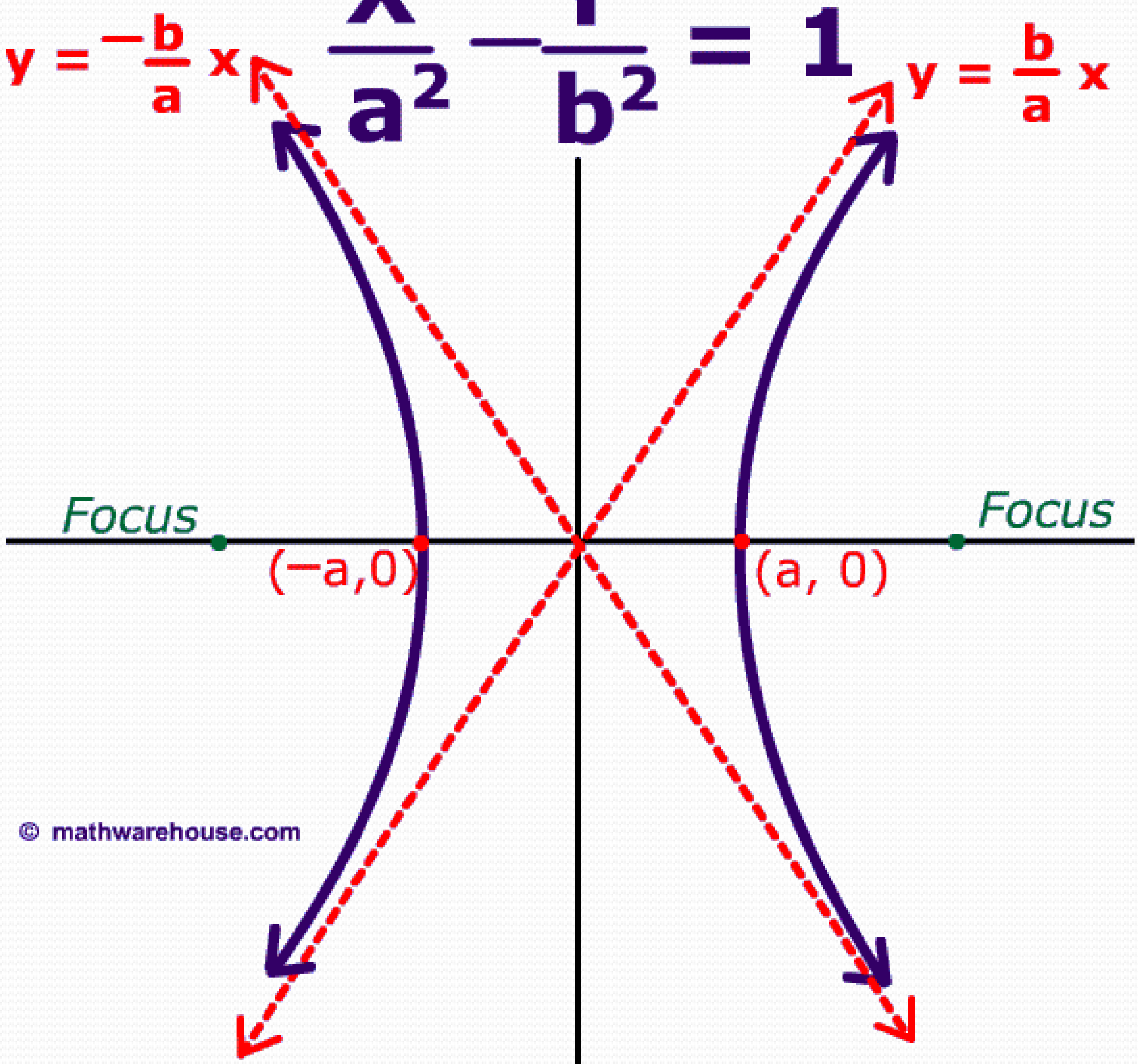

15
10
5
−10
−5
0
5
10
f(x) = 0.1 x2
f(x) = 0.3 x2
f(x) = 1.0 x2
f(x) = 3.0 x2

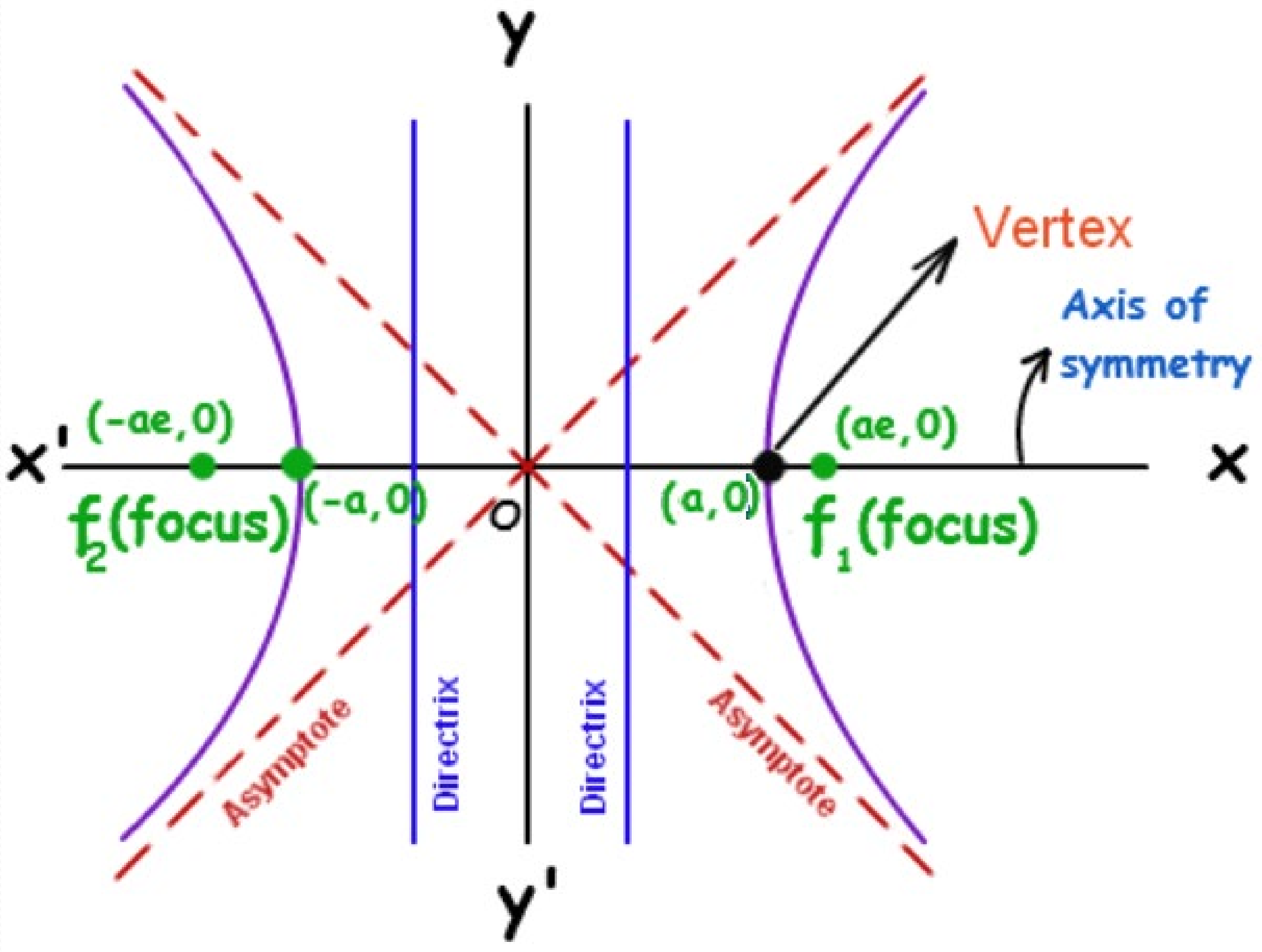


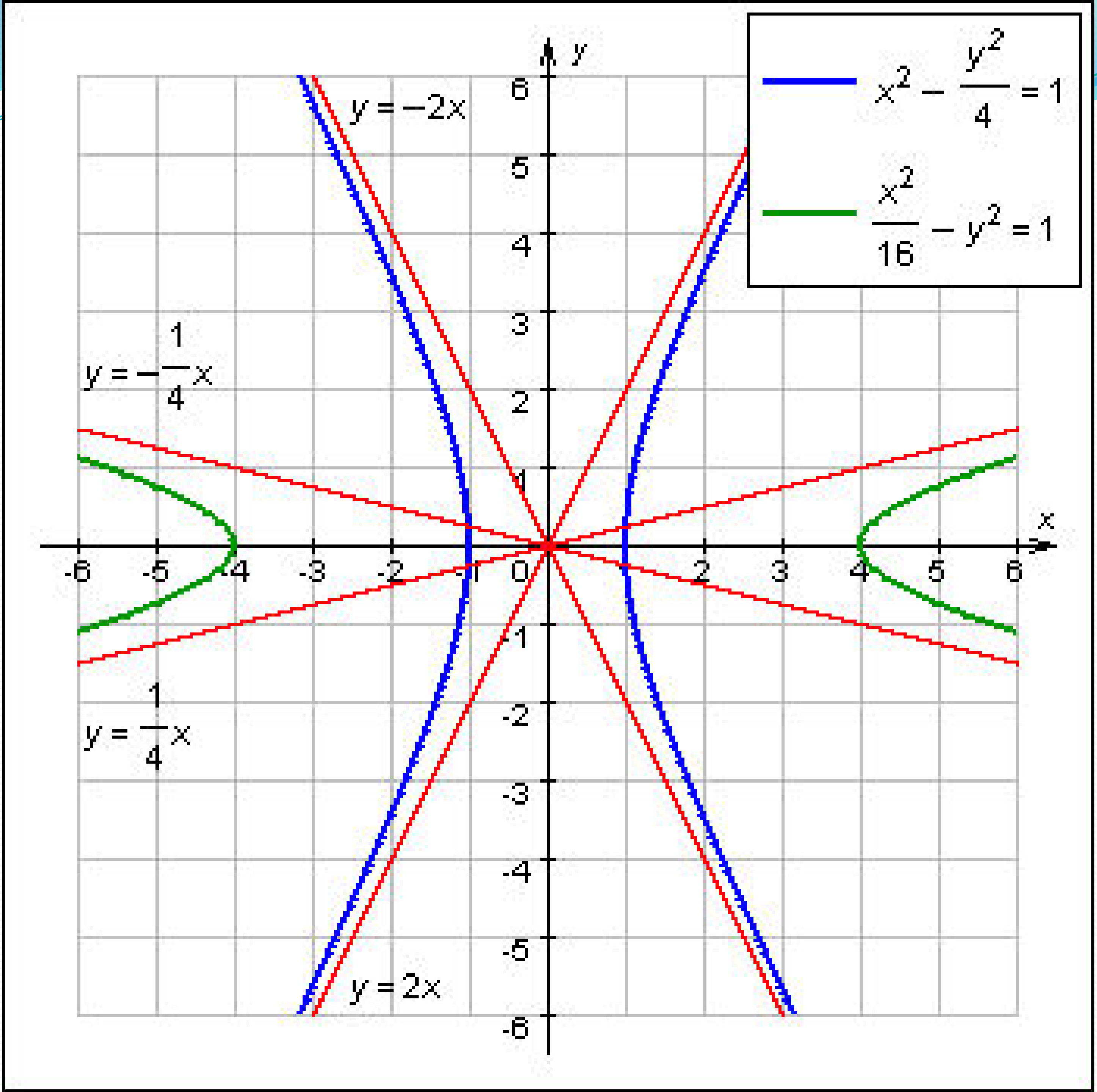
8
6
4
2
−2
−4
−6
−4
−2
0
2
4
6
f(x) = x(x−1)
f(x) = x(x−2)
f(x) = x(x−3)
f(x) = x(x−4)

Horizontal Transverse Axis

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



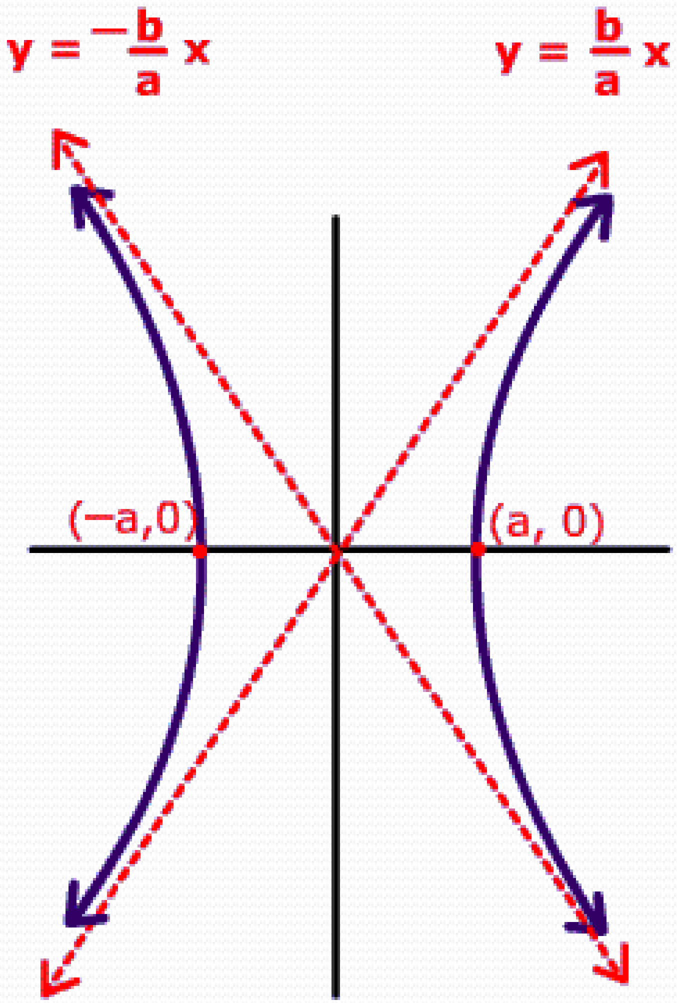

y
Vertex
Axis of symmetry
(-ae,0)
(ae,0)
x'
x
(-a,0)
(a,0)
O
f2(focus)
f1(focus)
Asymptote
Directrix
Directrix
Asymptote
y'



$x^2 - \frac{y^2}{4} = 1$
$\frac{x^2}{16} - y^2 = 1$
$y = -2x$
$y = 2x$
$y = -\frac{1}{4}x$
$y = \frac{1}{4}x$
x
y

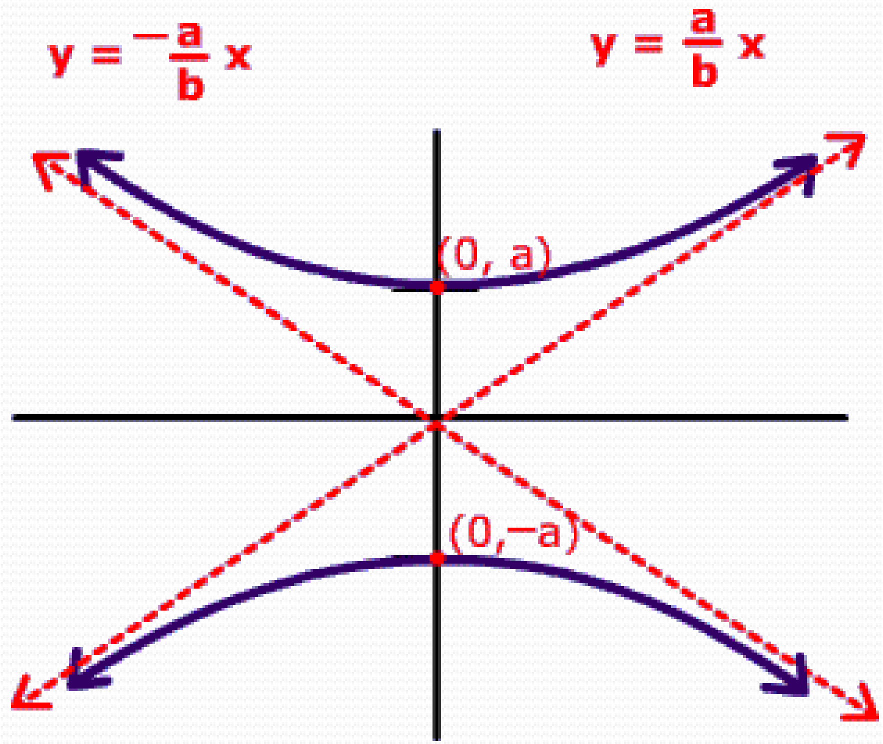
Horizontal Transverse Axis

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$



Vertical Transverse Axis

$$\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$$



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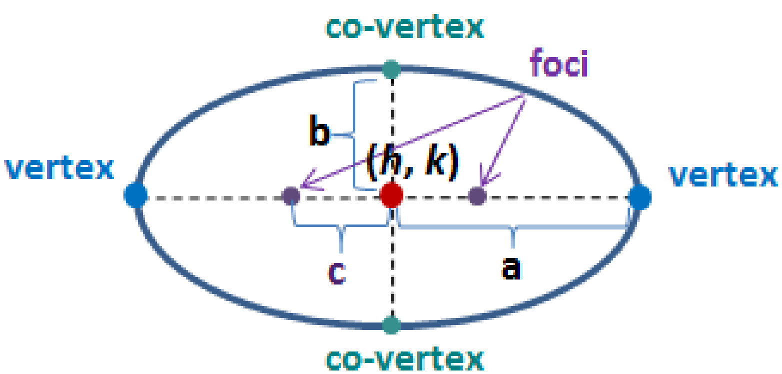
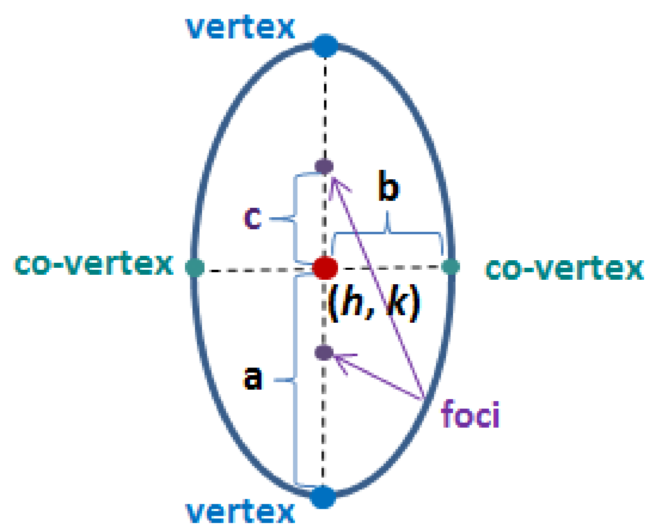
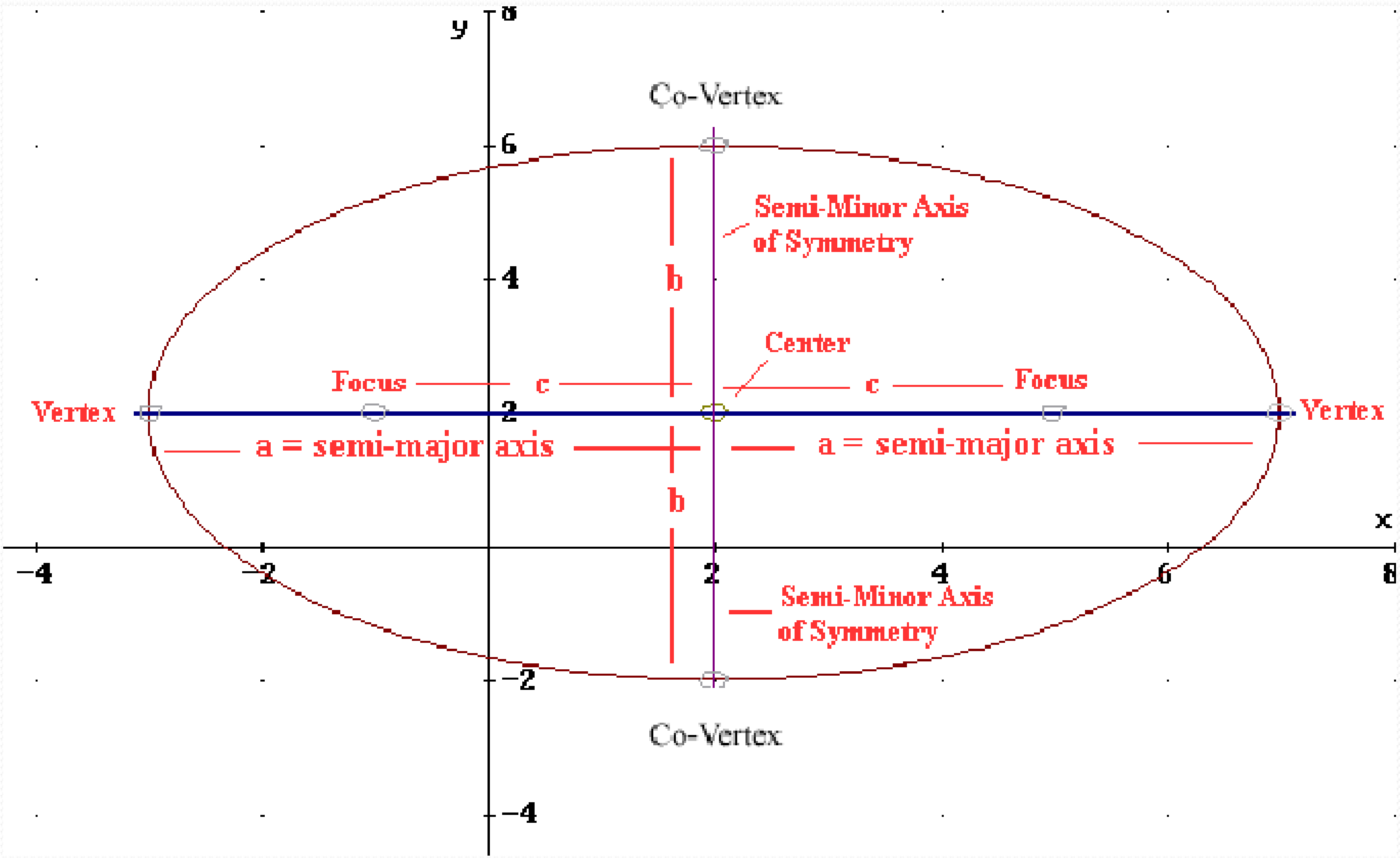
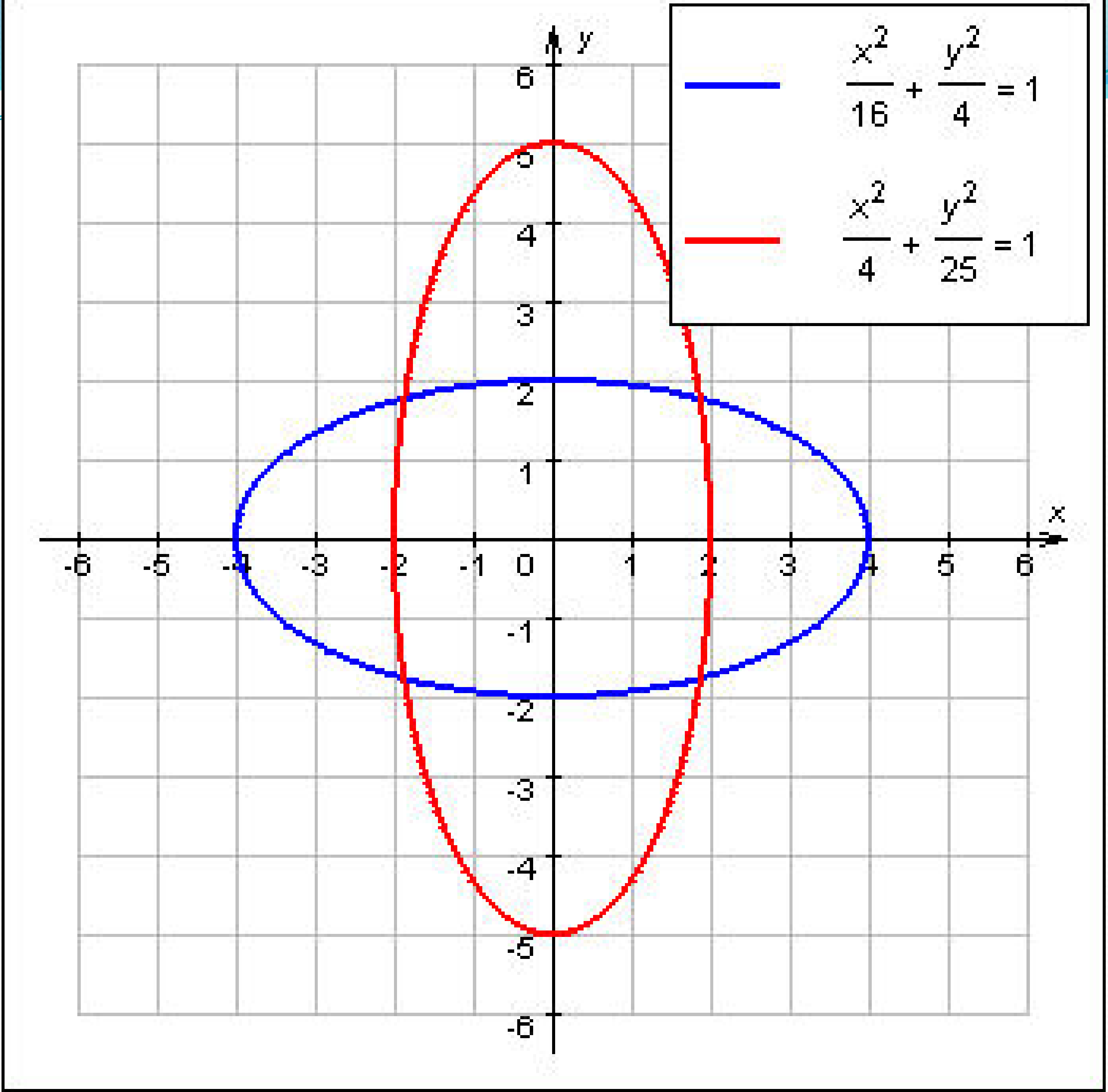
Horizontal Ellipse	**Vertical Ellipse**
At (0, 0): $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	At (0, 0): $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
General: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	**General:** $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$
$a^2-b^2=c^2$	$a^2-b^2=c^2$
Center: (h, k) **Foci:** $(h\pm c, k)$	**Center:** (h, k) **Foci:** $(h, k\pm c)$
Vertices: $(h\pm a, k)$ **Co-Vertices:** $(h, k\pm b)$	**Vertices:** $(h, k\pm a)$ **Co-Vertices:** $(h\pm b, k)$



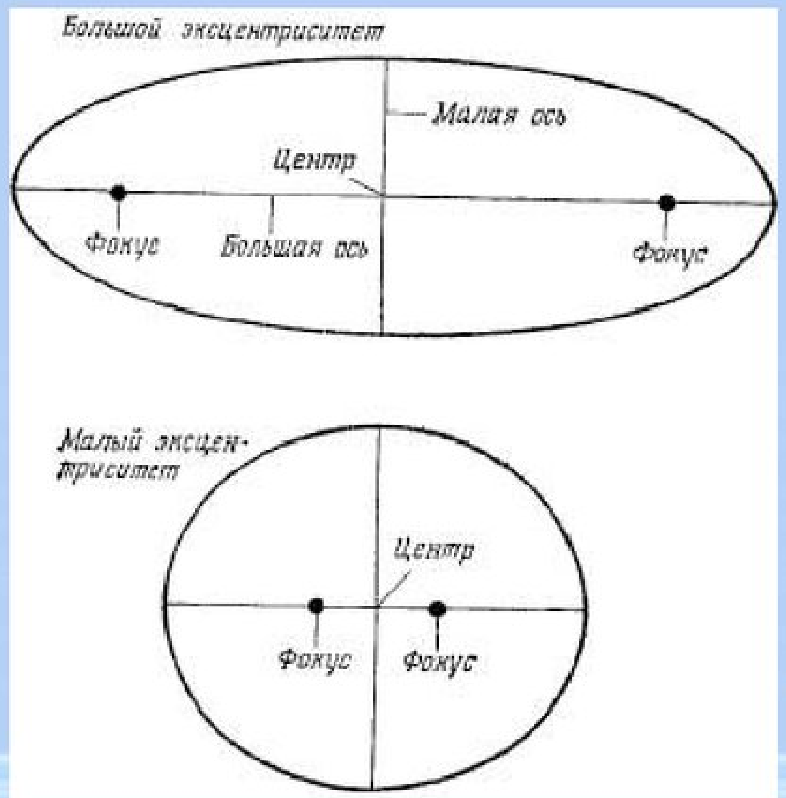


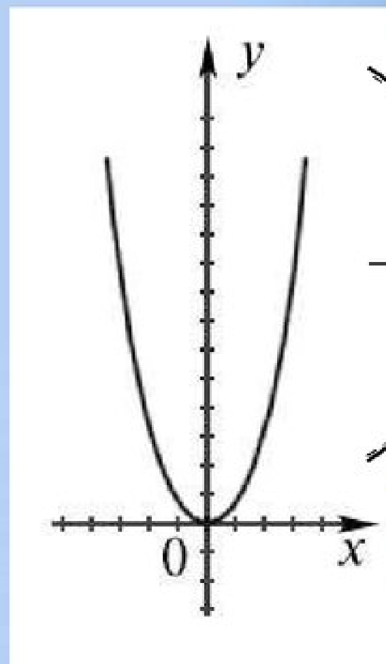


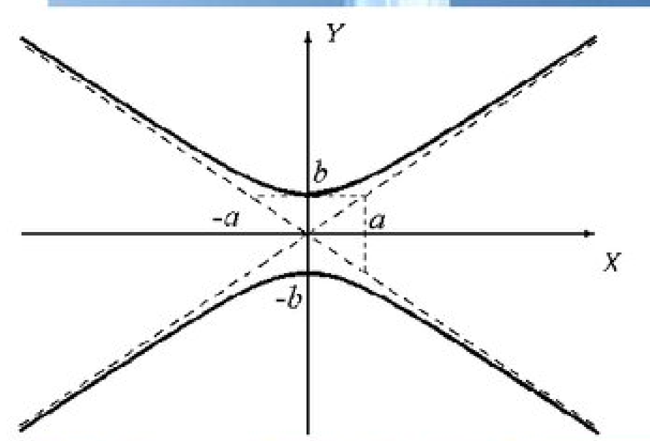
y
x
Co-Vertex
Semi-Minor Axis
of Symmetry
b
Center
Focus
c
c
Focus
Vertex
Vertex
a = semi-major axis
a = semi-major axis
b
Semi-Minor Axis
of Symmetry
Co-Vertex
−4
−2
2
4
6
8
6
4
2
−2
−4

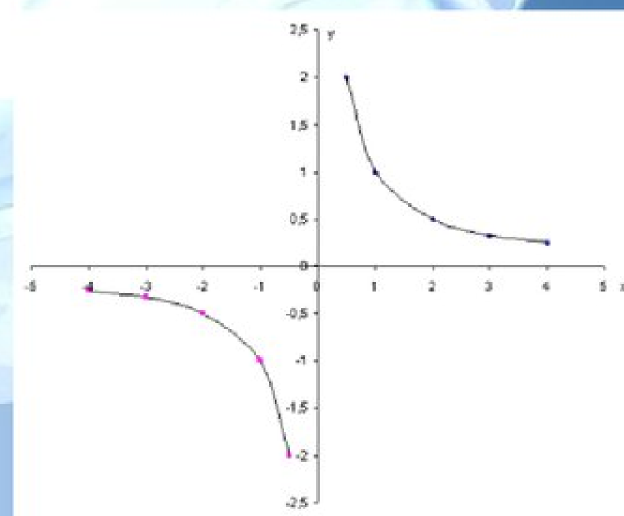

y
x
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
$\frac{x^2}{16} + \frac{y^2}{4} = 1$
$\frac{x^2}{4} + \frac{y^2}{25} = 1$

Кривые второго порядка









$$x^2+a_1y^2+a_2xy+a_3x+a_4y+a_5=0.$$

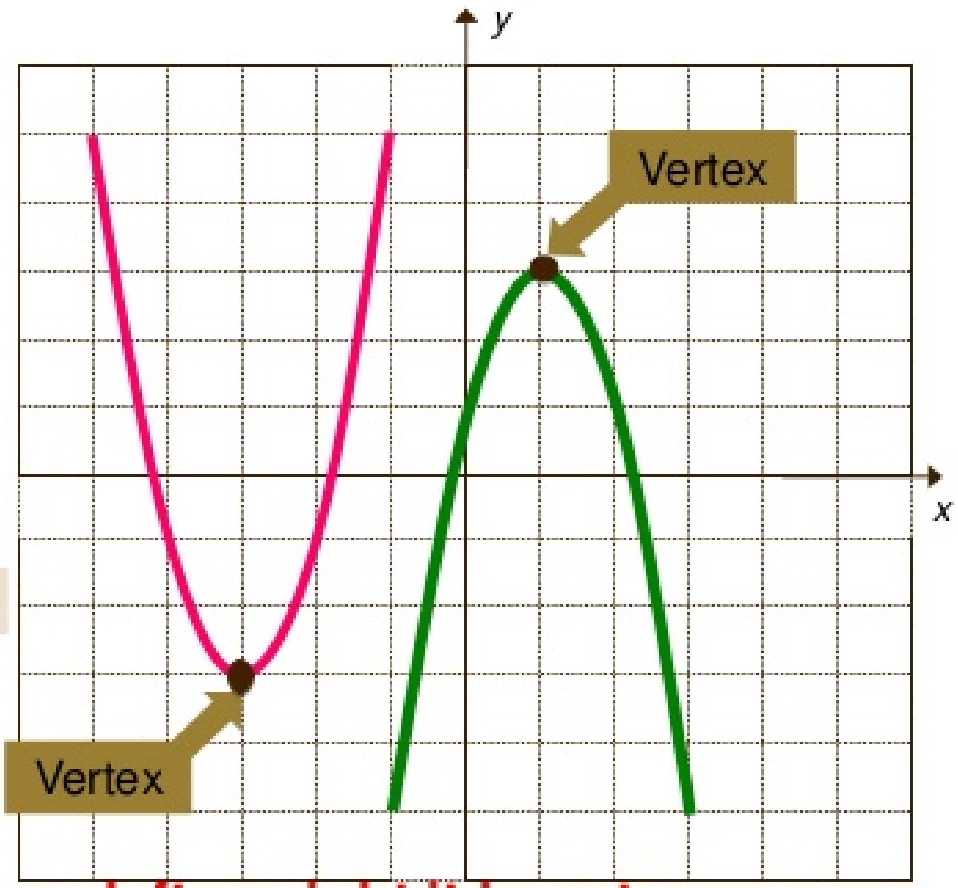
Quadratic Functions

The graph of a quadratic function is a **parabola**

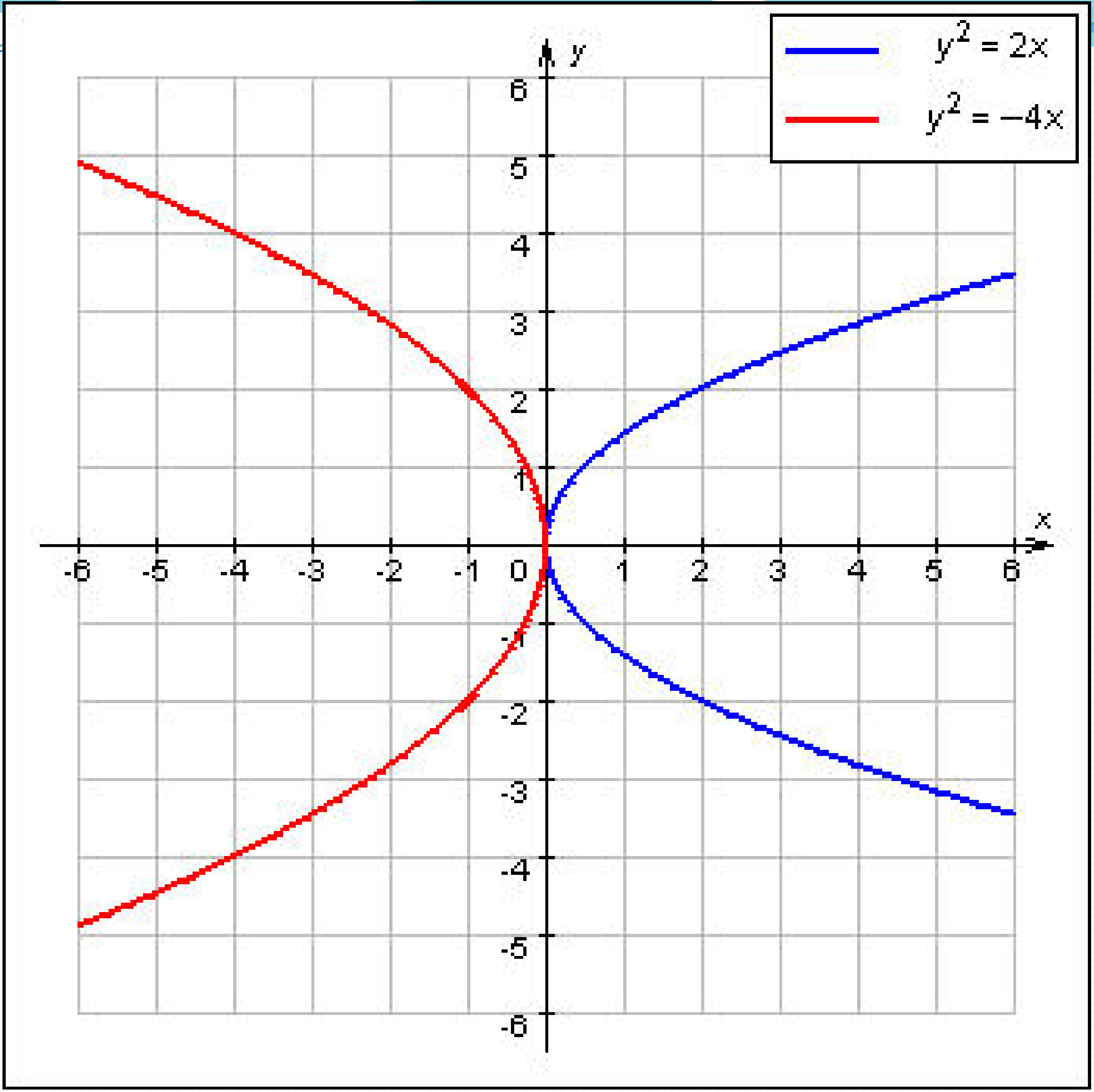
A parabola can open up or down.

If the parabola opens up, the lowest point is called the vertex (**minimum**).

If the parabola opens down, the vertex is the highest point (**maximum**).

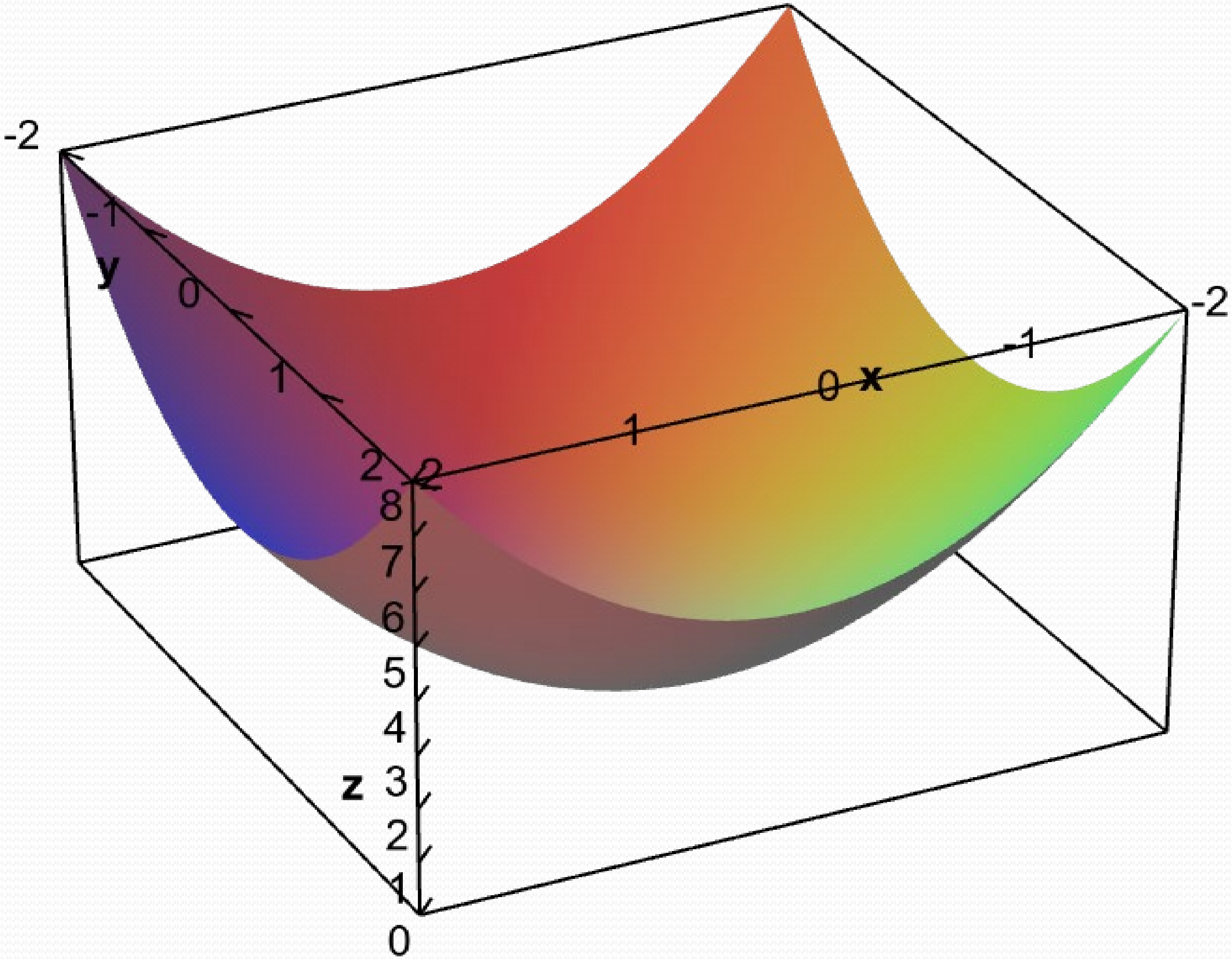


NOTE: if the parabola opens left or right it is not a function!

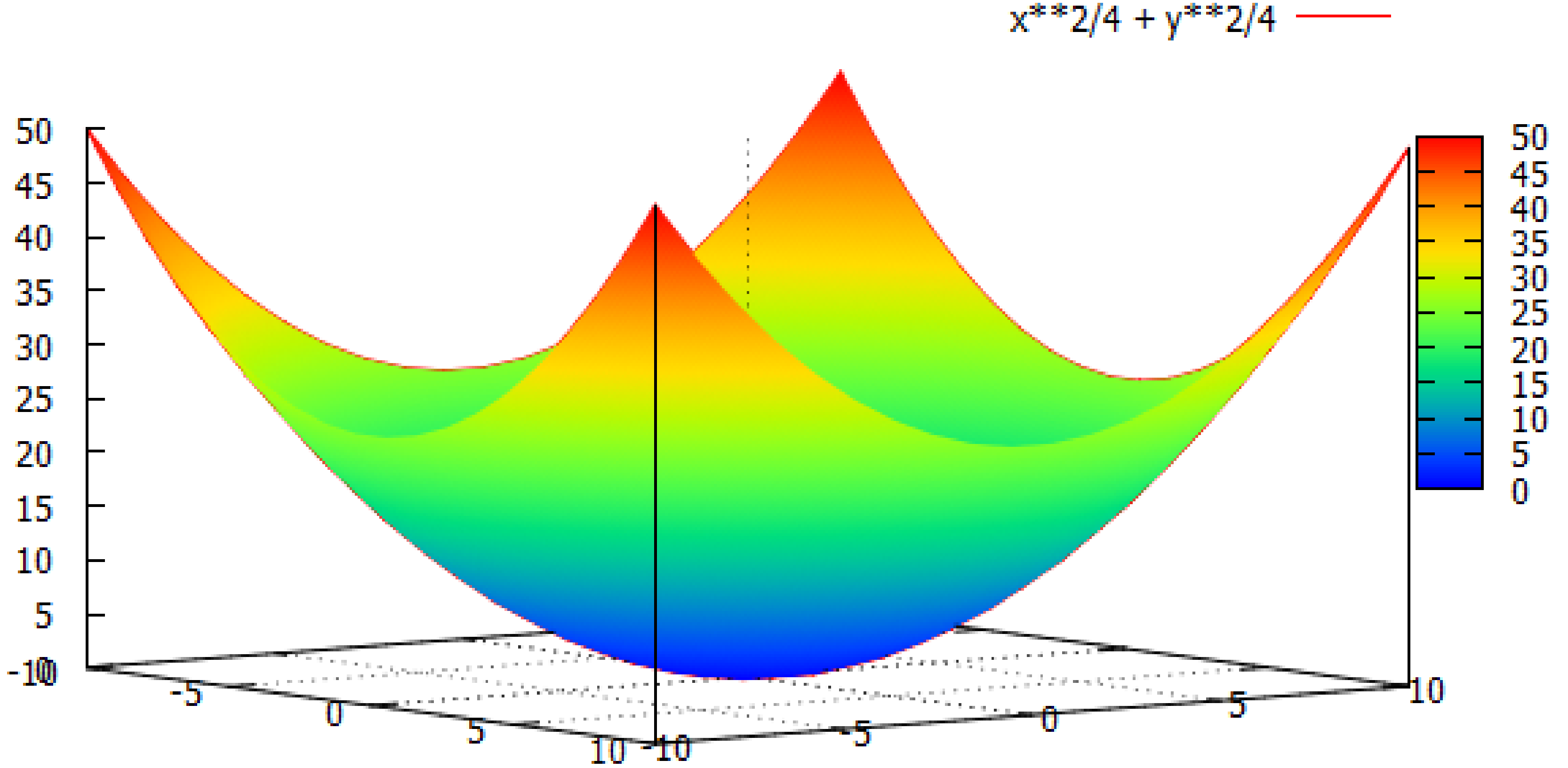


y
x
$y^2 = 2x$
$y^2 = -4x$
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6

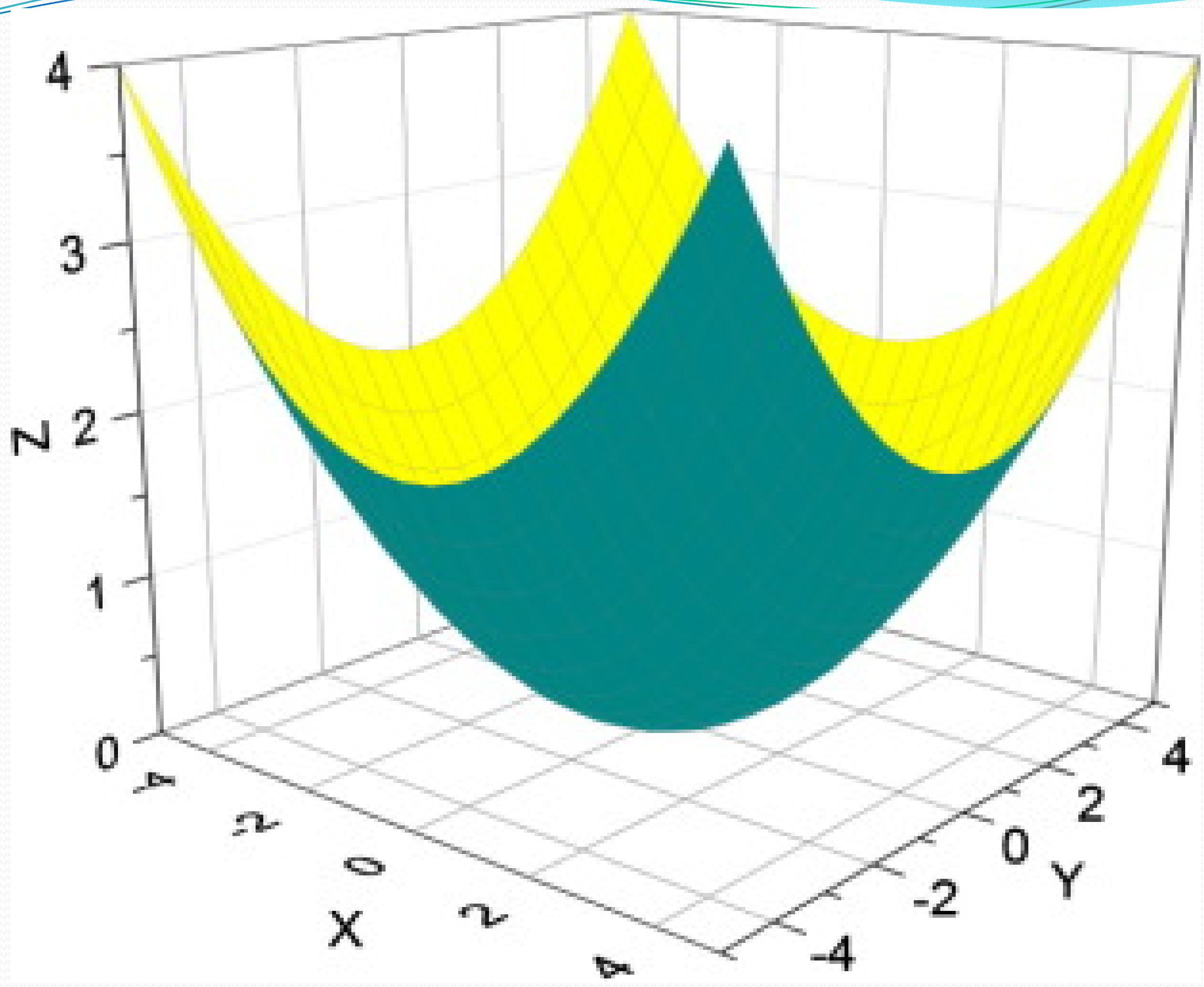
SURFACES in 3D



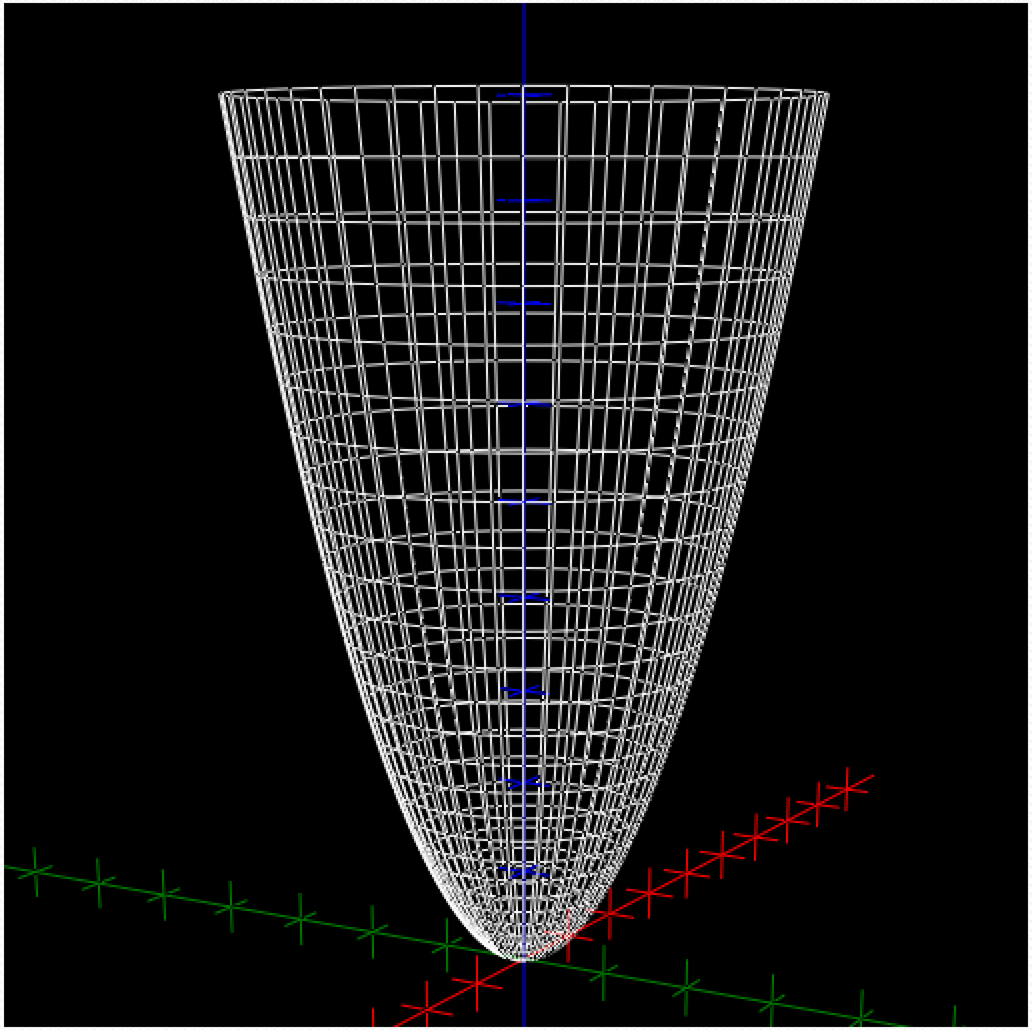
-2
-1
y
0
1
2
-2
-1
0
x
1
2
8
7
6
5
4
z
3
2
1
0



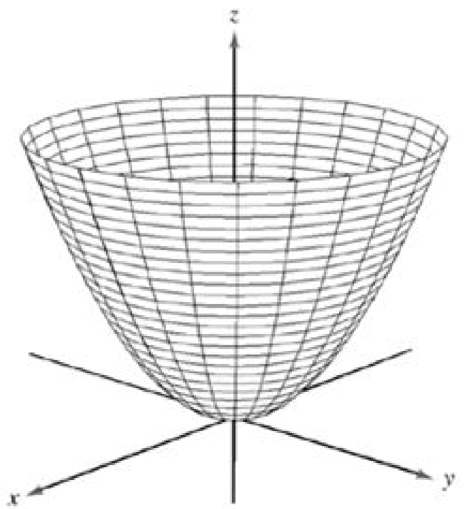
x**2/4 + y**2/4
50
45
40
35
30
25
20
15
10
5
0
-10
-5
0
5
10
-10
-5
0
5
10



Z
4
3
2
1
0
X
-4
-2
0
2
4
Y
4
2
0
-2
-4



Elliptic Paraboloid

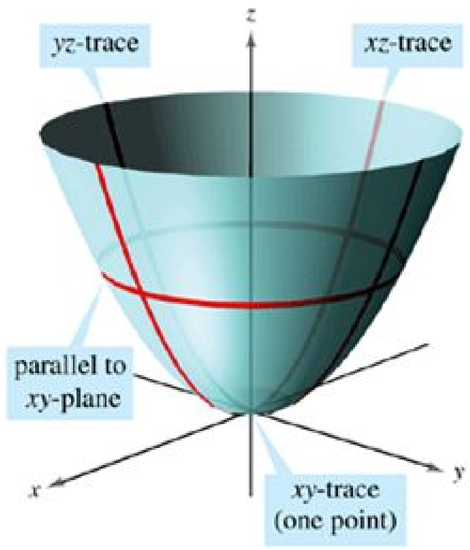


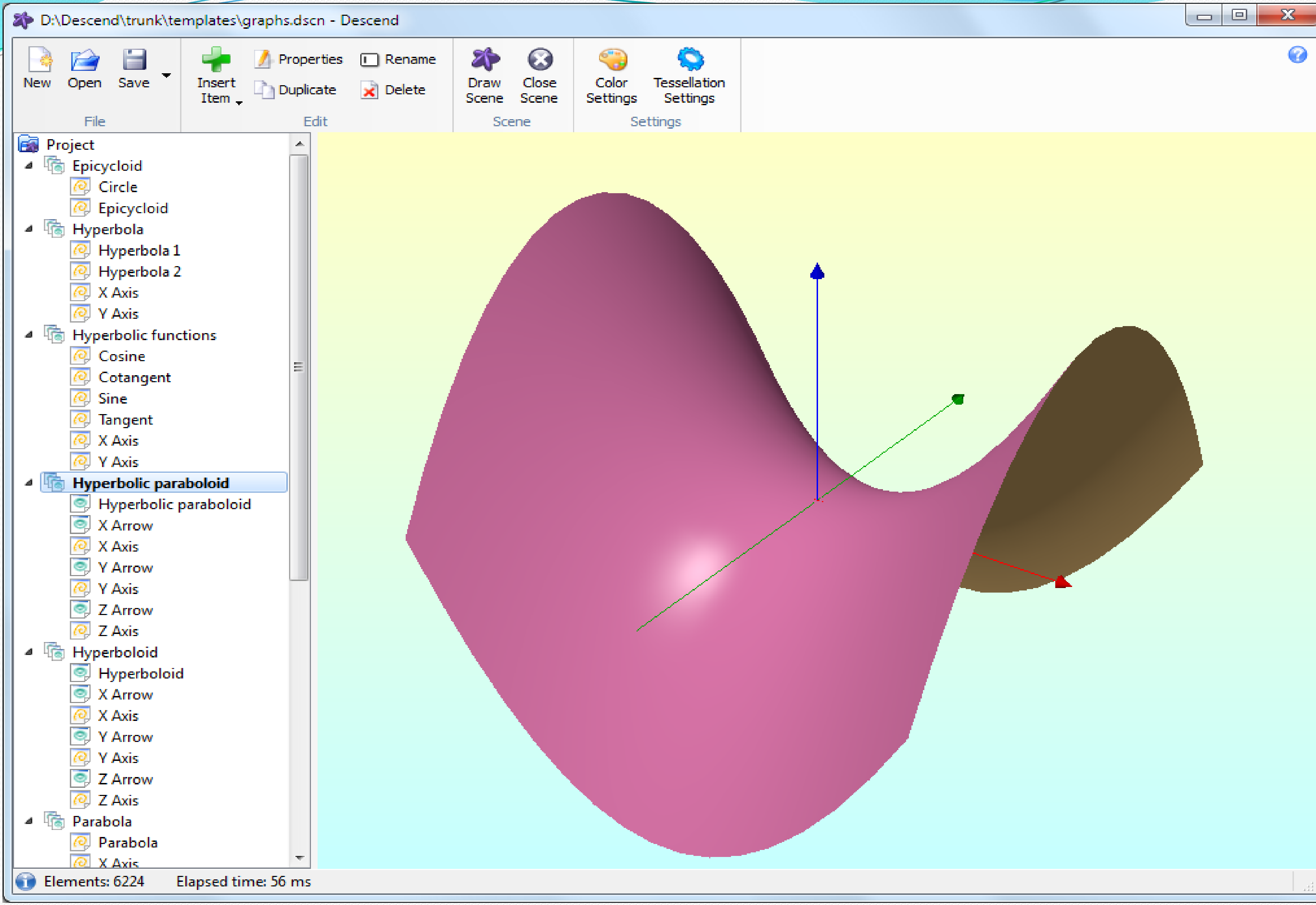
Elliptic Paraboloid

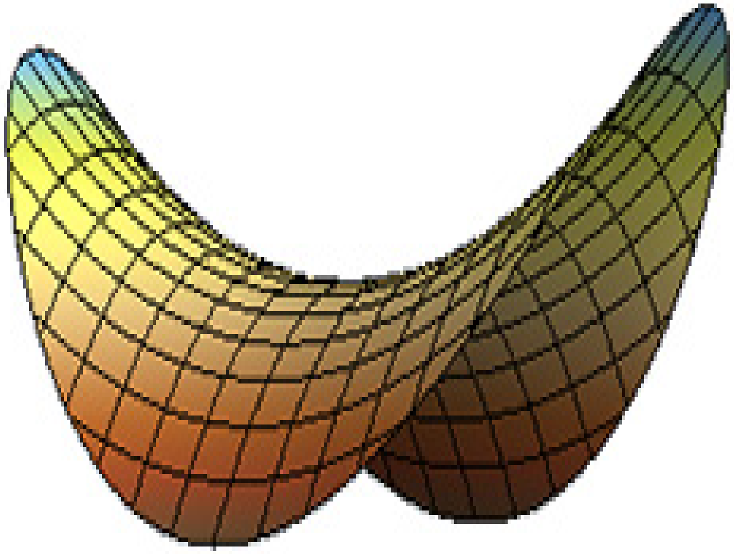
$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.

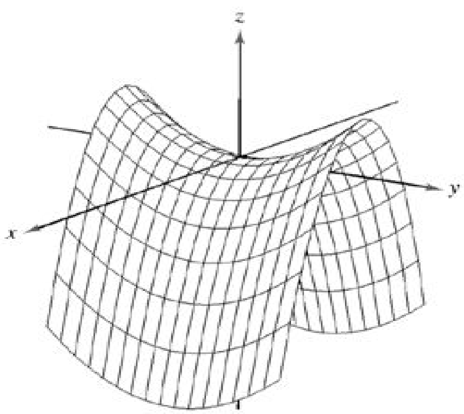



D:\Descend\trunk\templates\graphs.dscn - Descend
New
Open
Save
File
Insert Item
Properties
Rename
Duplicate
Delete
Edit
Draw Scene
Close Scene
Scene
Color Settings
Tessellation Settings
Settings
Project
Epicycloid
Circle
Epicycloid
Hyperbola
Hyperbola 1
Hyperbola 2
X Axis
Y Axis
Hyperbolic functions
Cosine
Cotangent
Sine
Tangent
X Axis
Y Axis
Hyperbolic paraboloid
Hyperbolic paraboloid
X Arrow
X Axis
Y Arrow
Y Axis
Z Arrow
Z Axis
Hyperboloid
Hyperboloid
X Arrow
X Axis
Y Arrow
Y Axis
Z Arrow
Z Axis
Parabola
Parabola
Elements: 6224
Elapsed time: 56 ms



$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$$

Hyperbolic Paraboloid

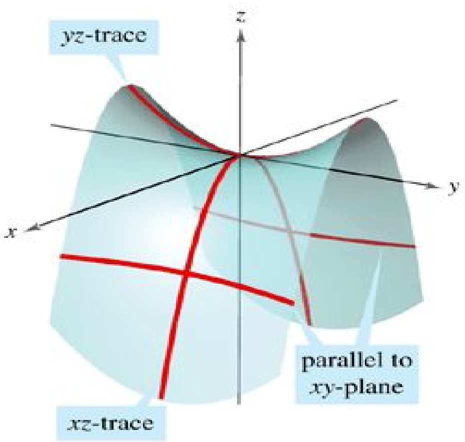


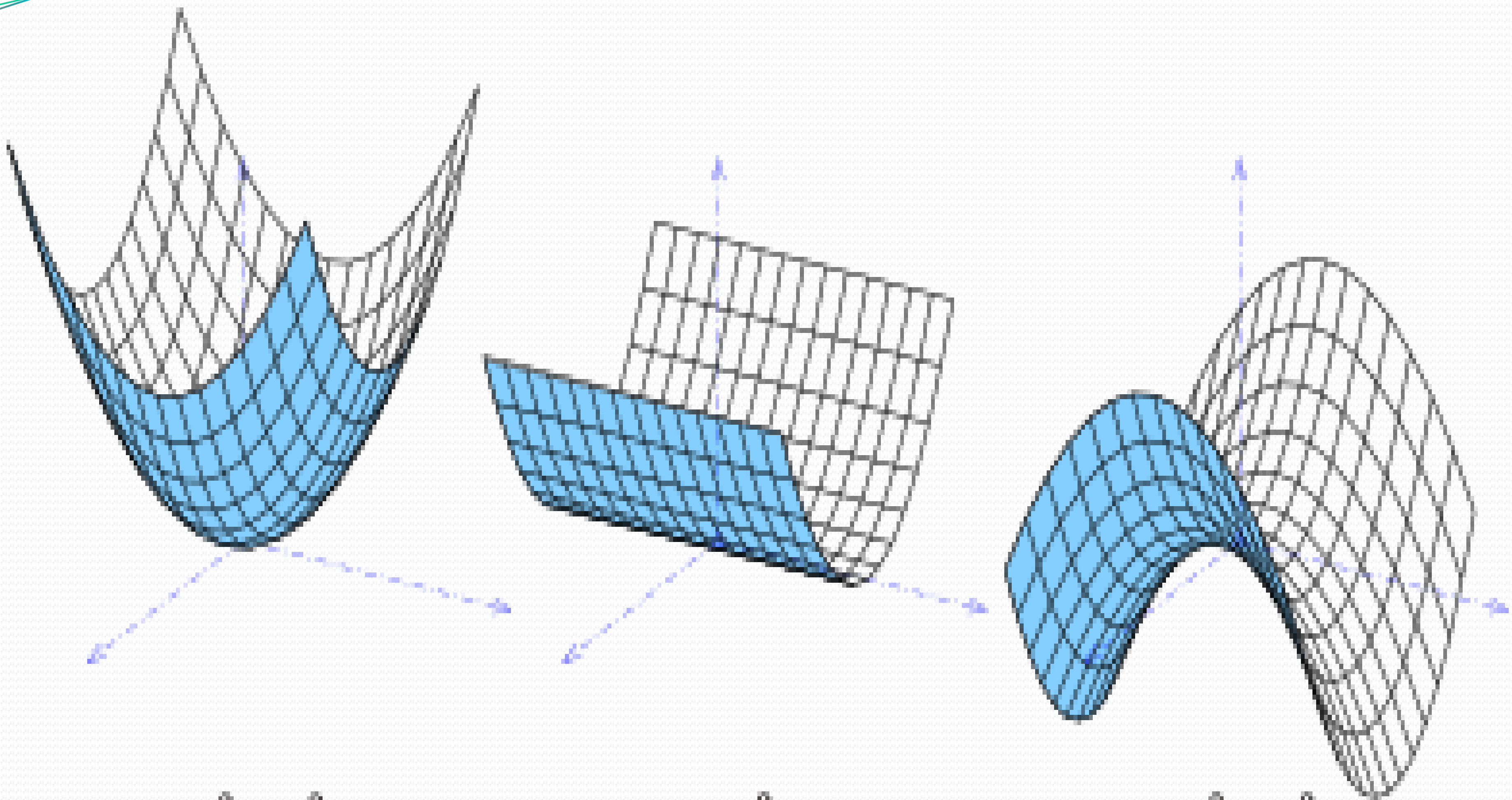
Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.





$z = x^2 + y^2$ $z = x^2$ $z = x^2 - y^2$

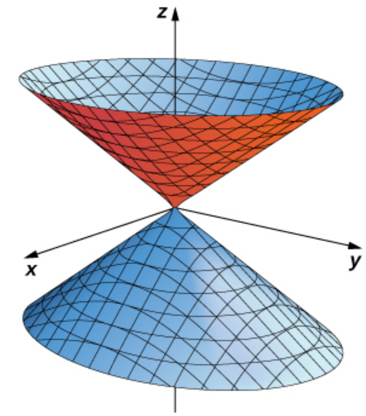
Characteristics of Common Quadric Surfaces

Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

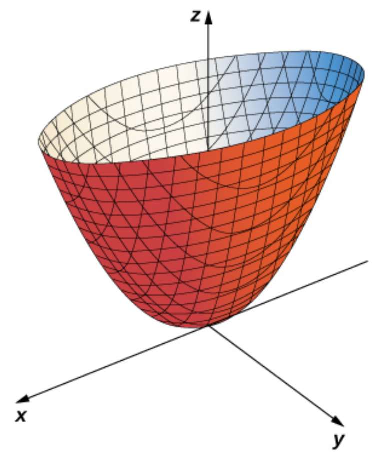


Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.

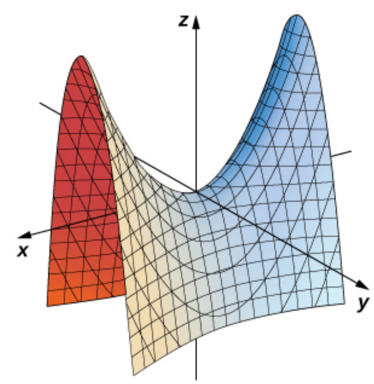


Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces
In plane $z = p$: a hyperbola
In plane $y = q$: a parabola
In plane $x = r$: a parabola

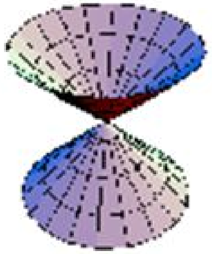
The axis of the surface corresponds to the linear variable.



Quadric surfaces

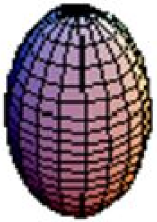
- Double cones

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$



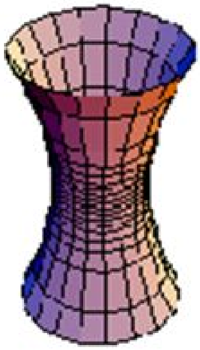
- Ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



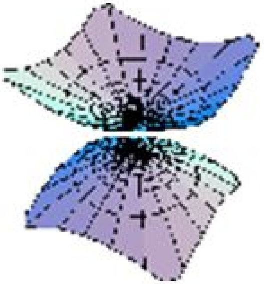
- Hyperboloids of one sheet

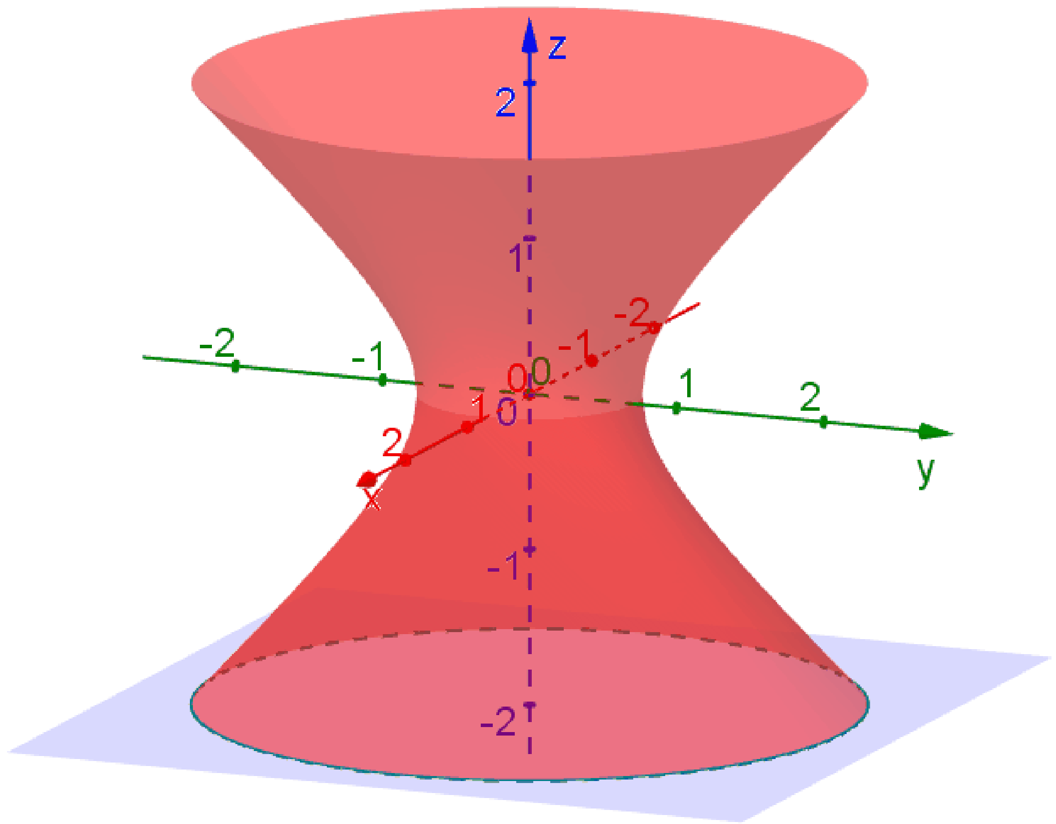
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

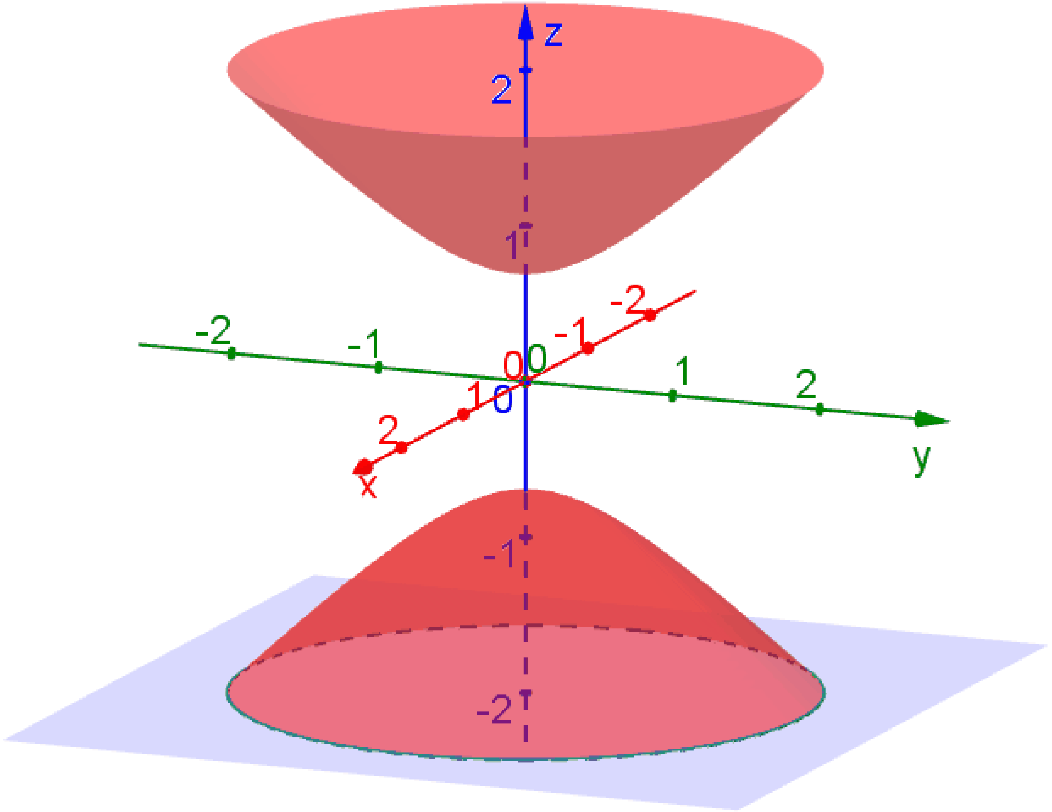


- Hyperboloids of two sheets

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$




z
2
1
-1
-2
y
x
-2
-1
0
1
2



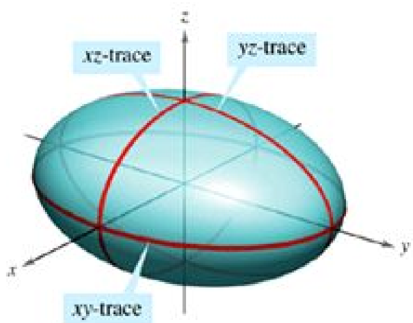
z
2
1
-1
-2
0
y
-2
-1
1
2
x

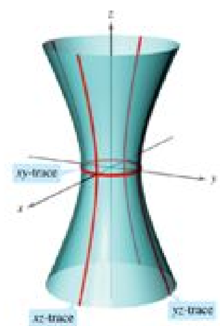
Quadric Surface

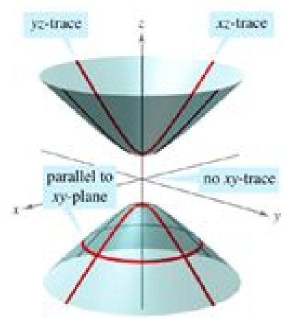
The equation of a **quadric surface** in space is a second-degree equation of the form

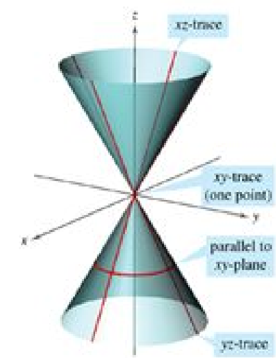
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

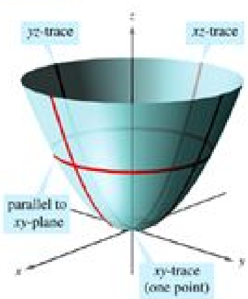
There are six basic types of quadric surfaces: **ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, and hyperbolic paraboloid.**

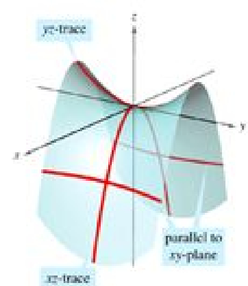


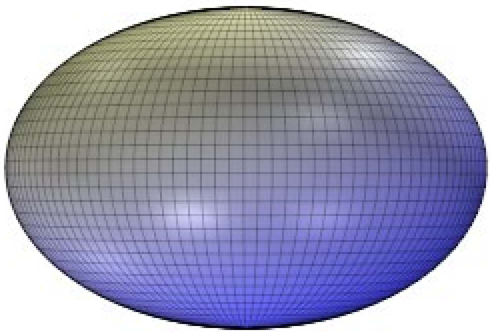




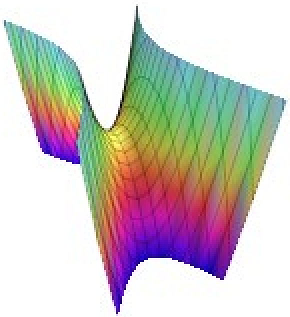




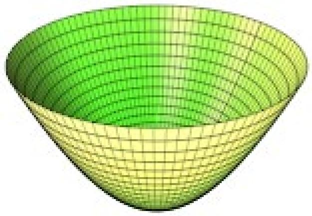




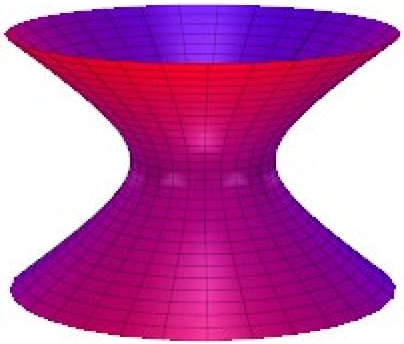
Ellipsoid



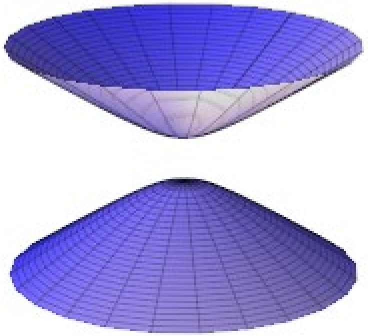
Hyperbolic paraboloid



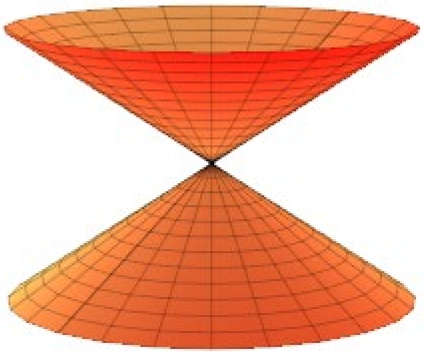
Elliptic paraboloid



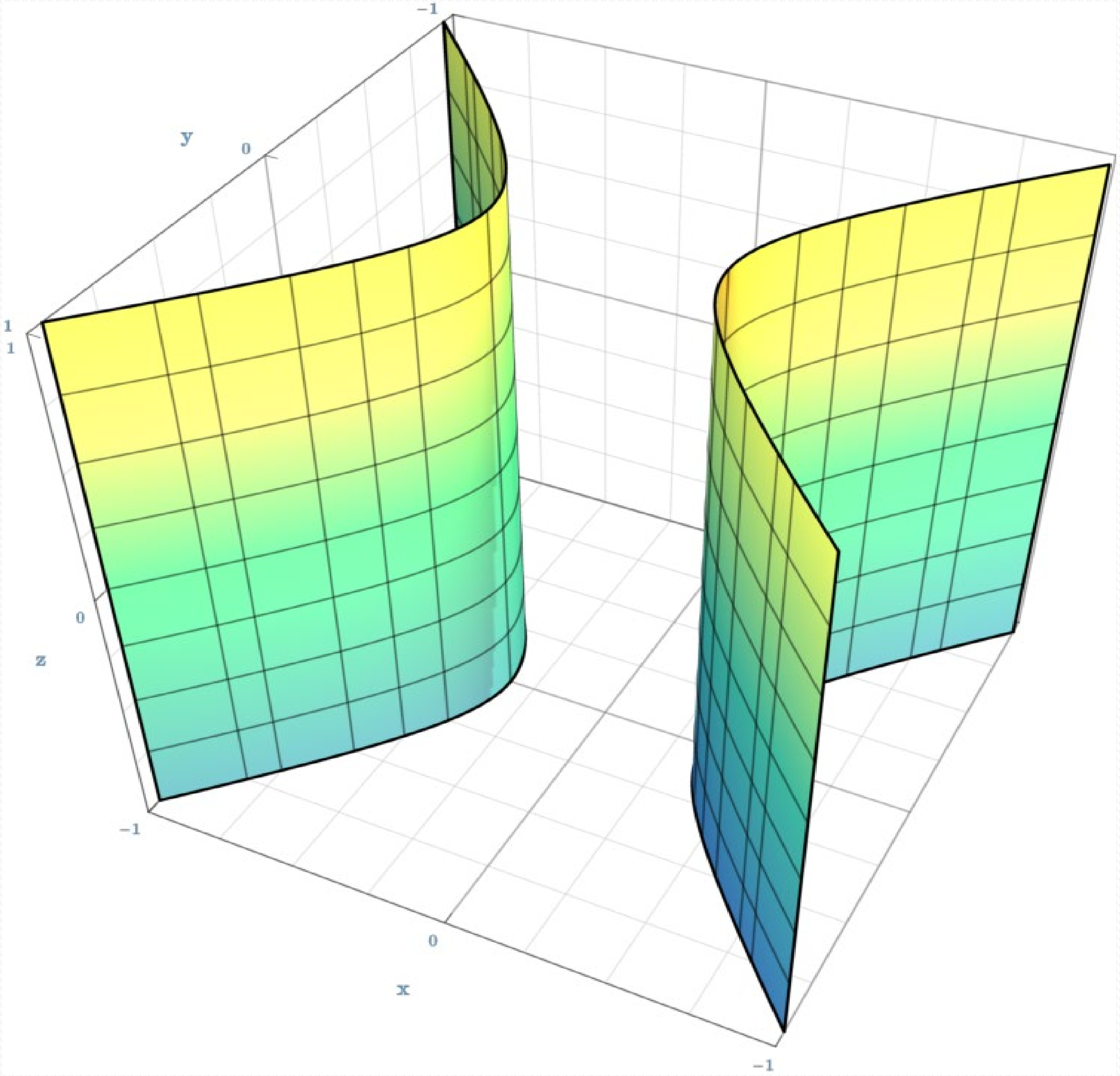
Hyperboloid of one sheet



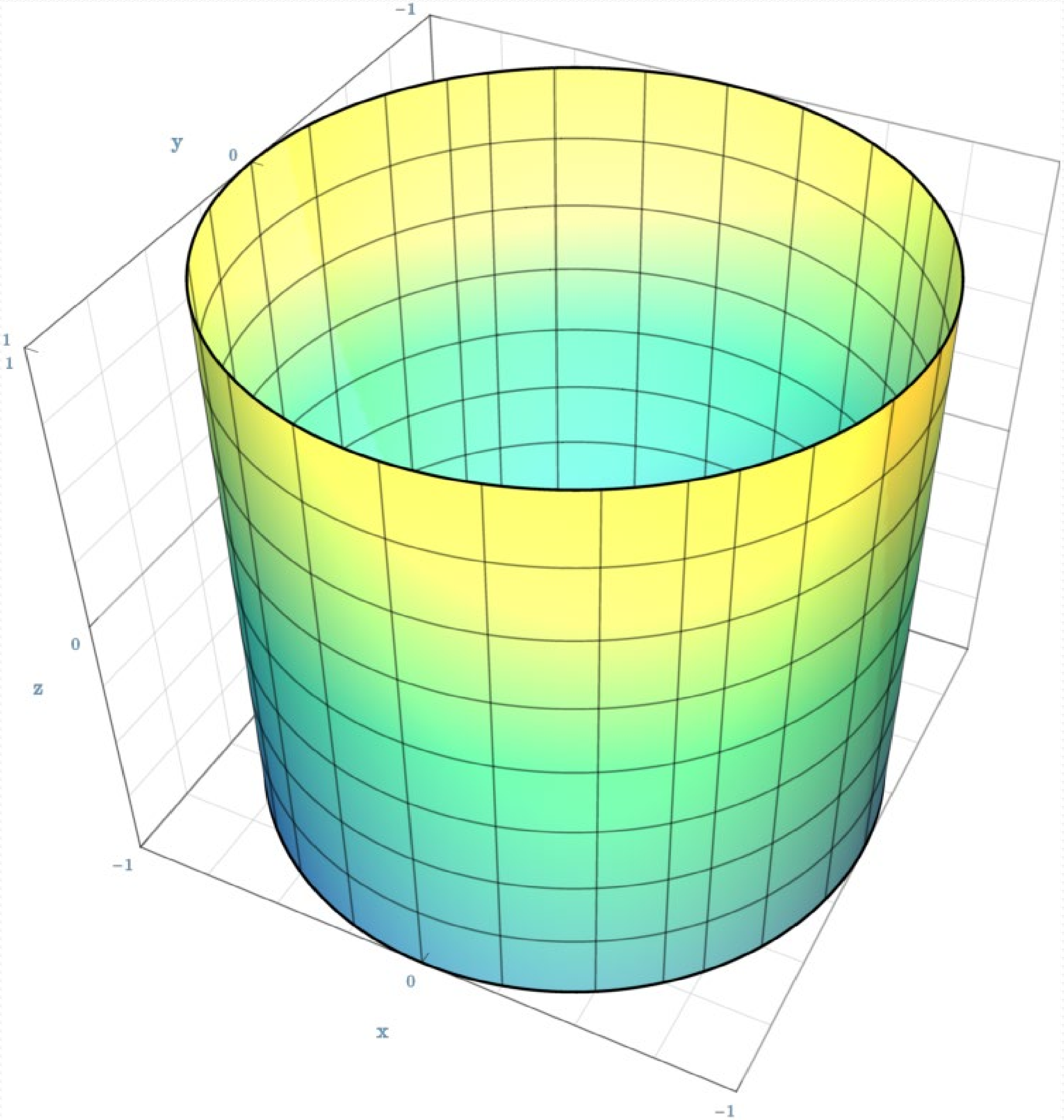
Hyperboloid of two sheets



Cone



−1
y
0
1
1
0
z
−1
0
x
−1



−1
y
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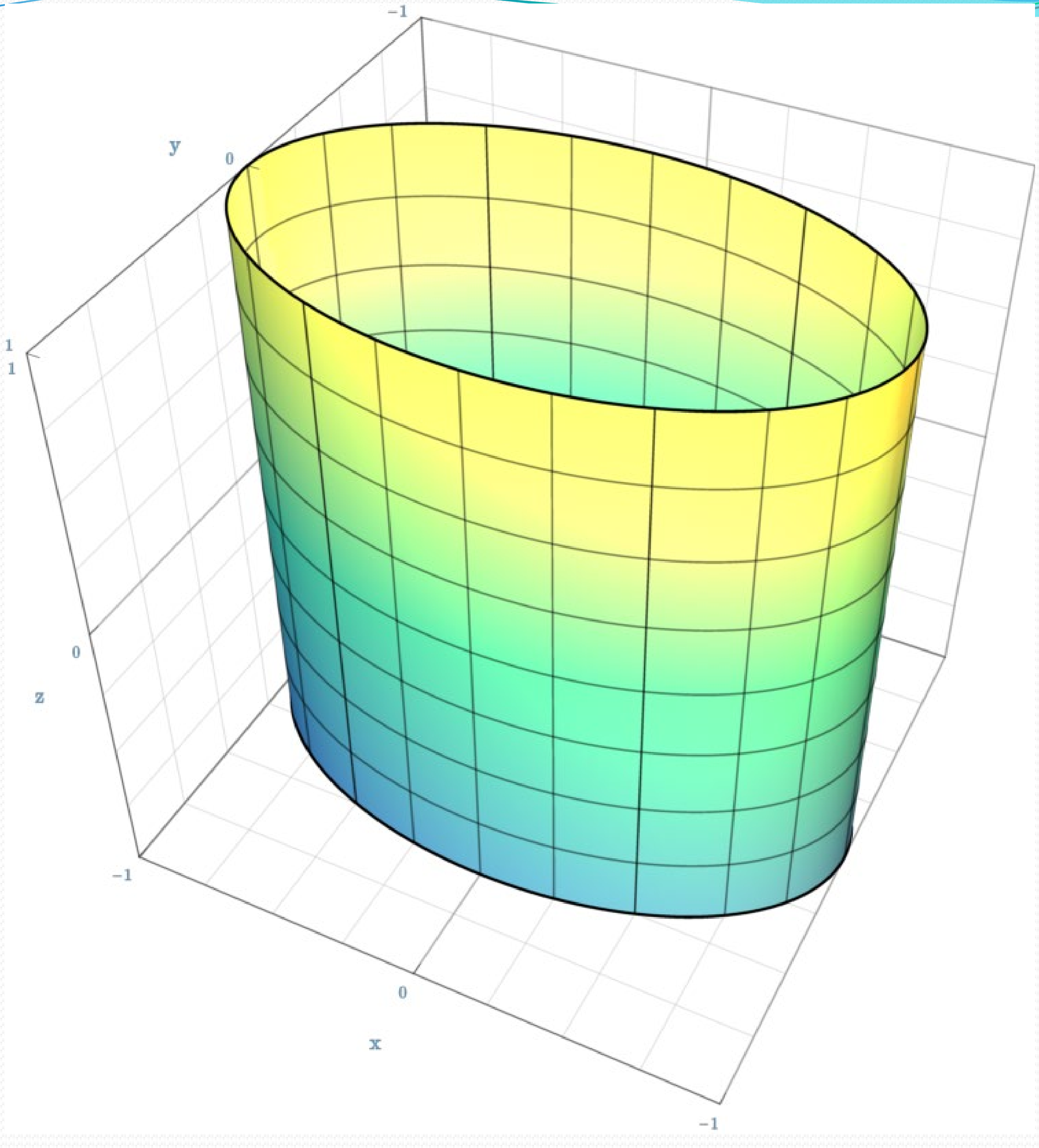
x
−1



−1
y
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z
−1
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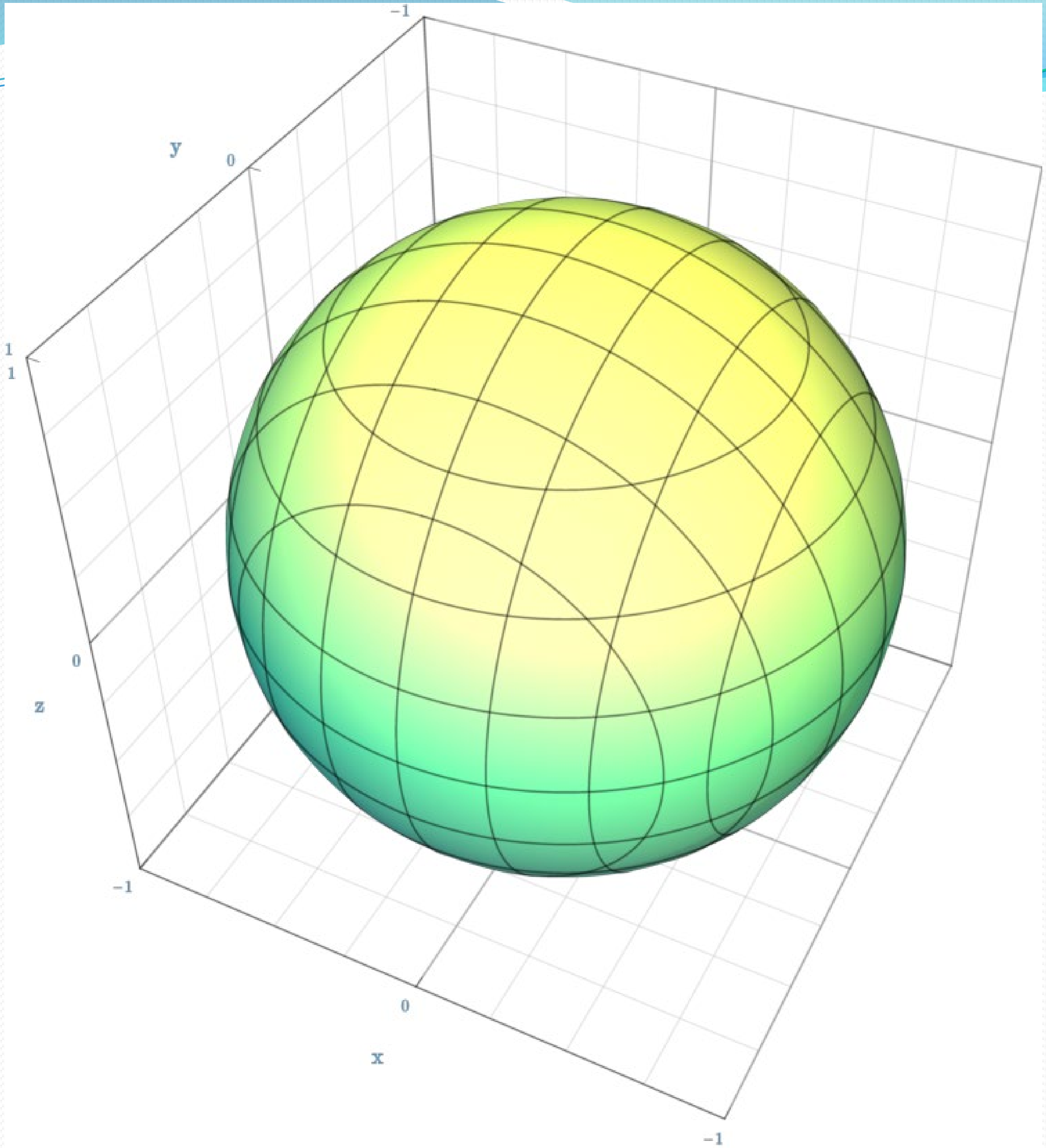
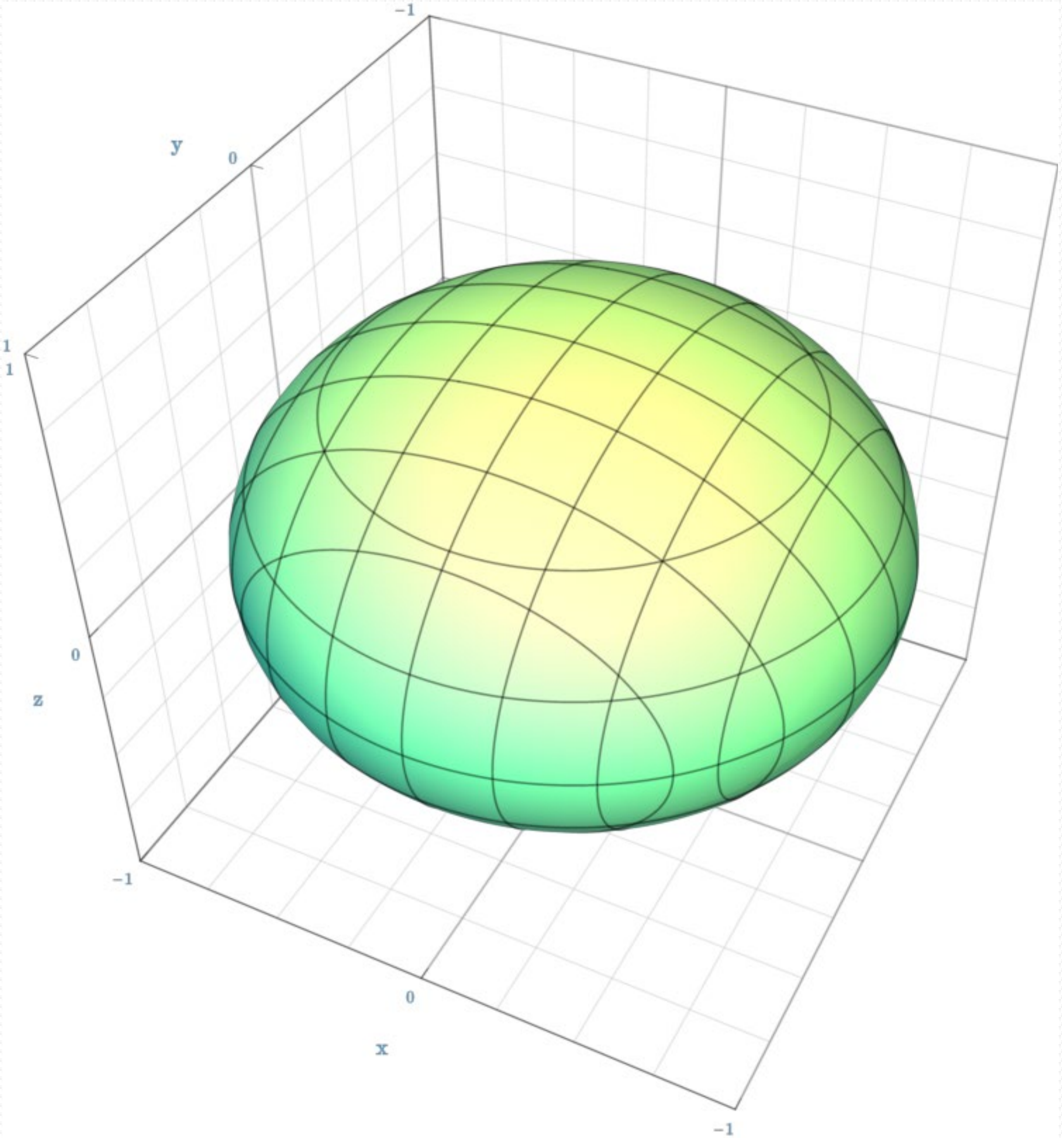
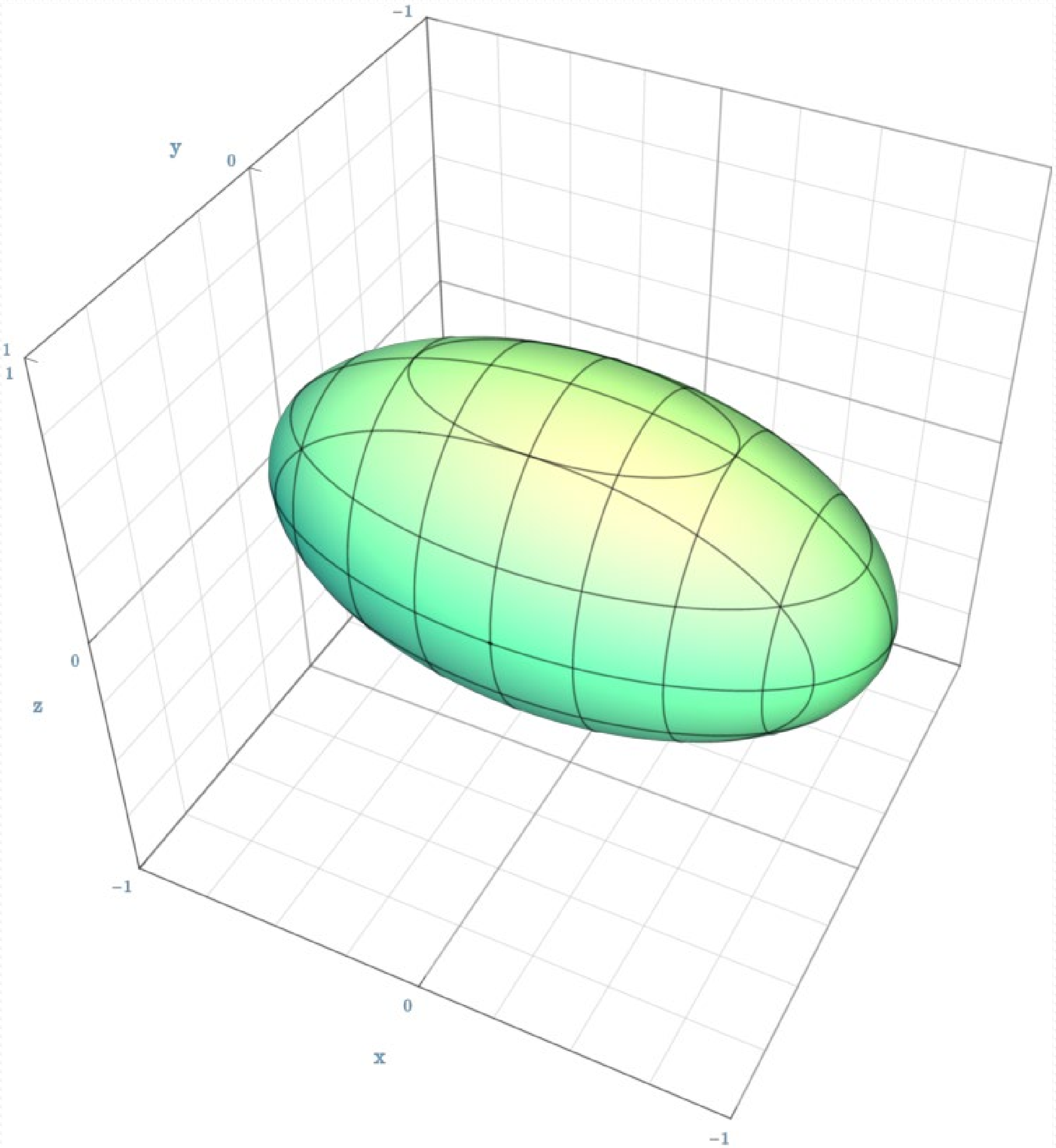
x
−1

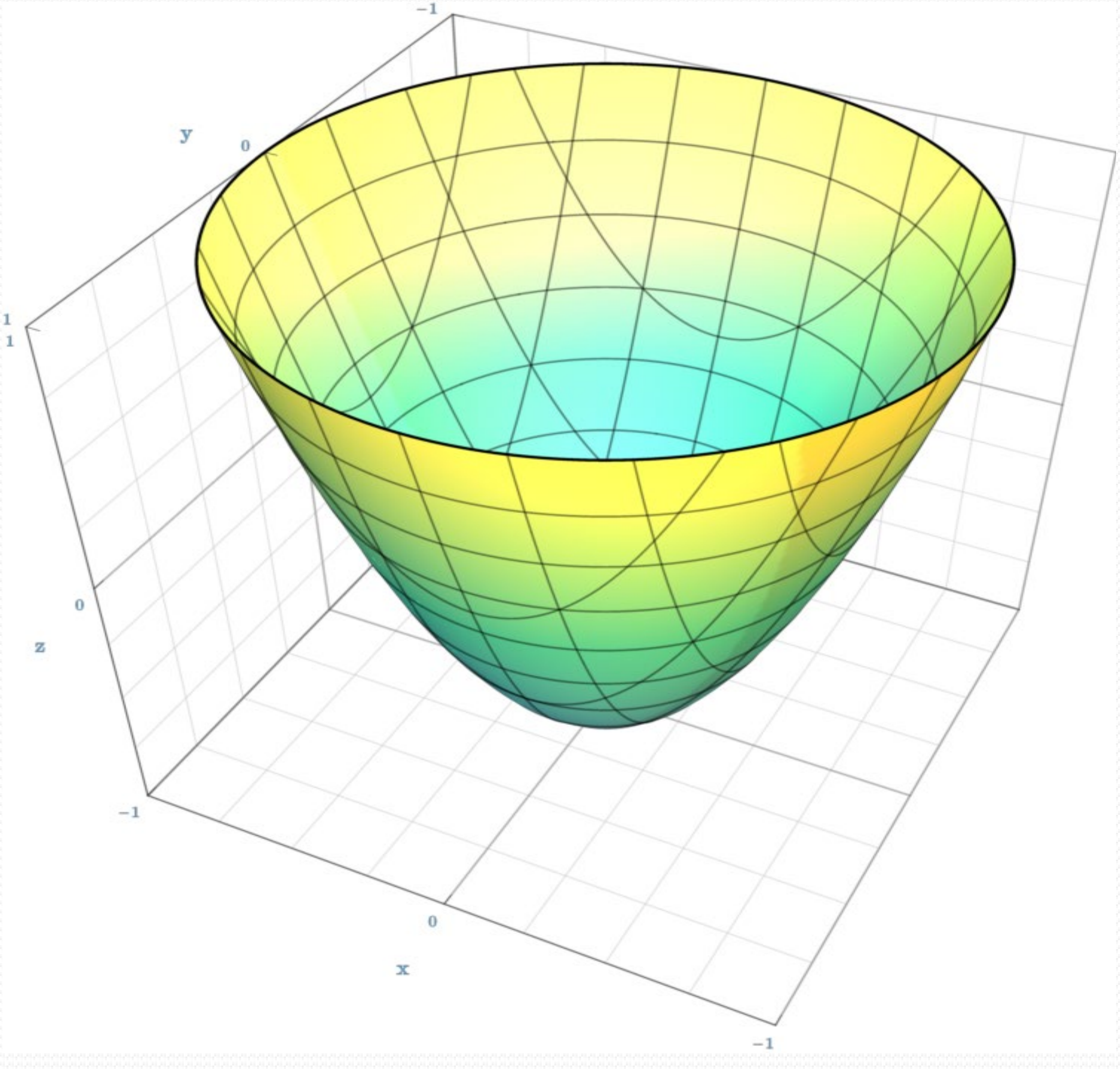


−1
y
0
1
1
0
z
−1
0
x
−1

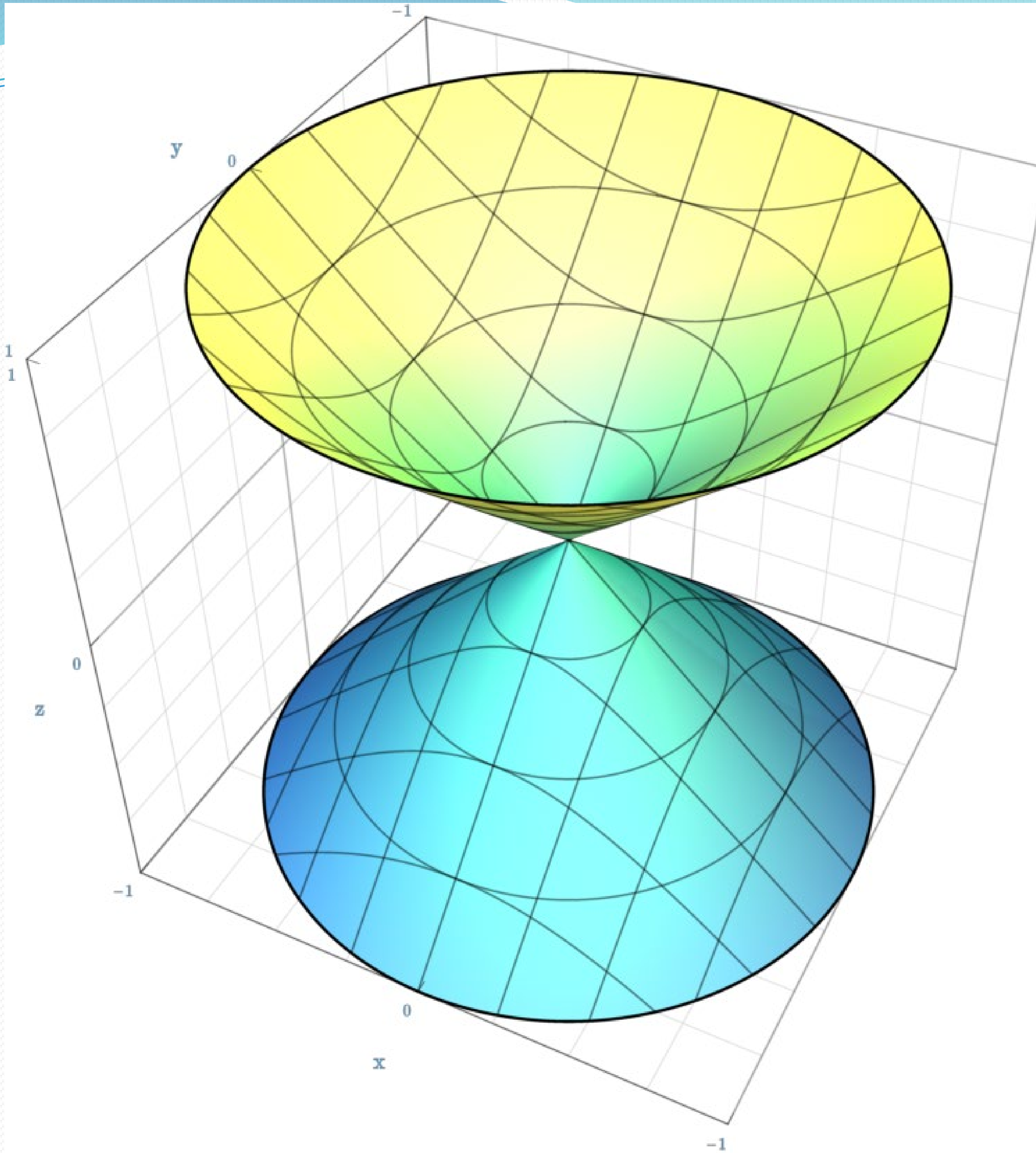

−1
y
0
1
1
0
z
−1
0
x
−1



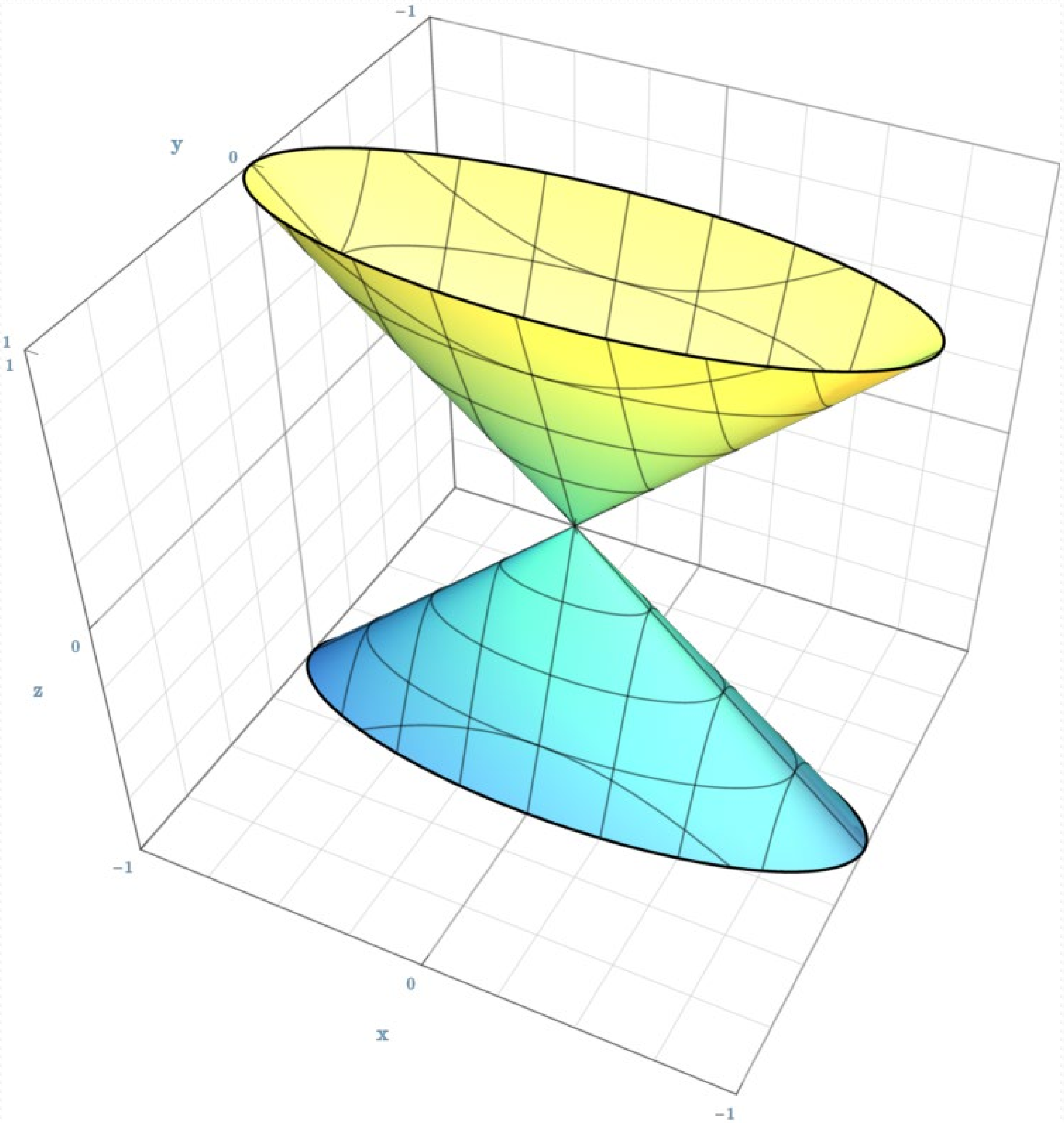
−1
y
0
1
1
0
z
−1
0
x
−1



−1
y
0
1
1
0
z
−1
0
x
−1



−1
y
0
1
1
0
z
−1
0
x
−1



−1
y
0
1
1
0
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−1
0
x
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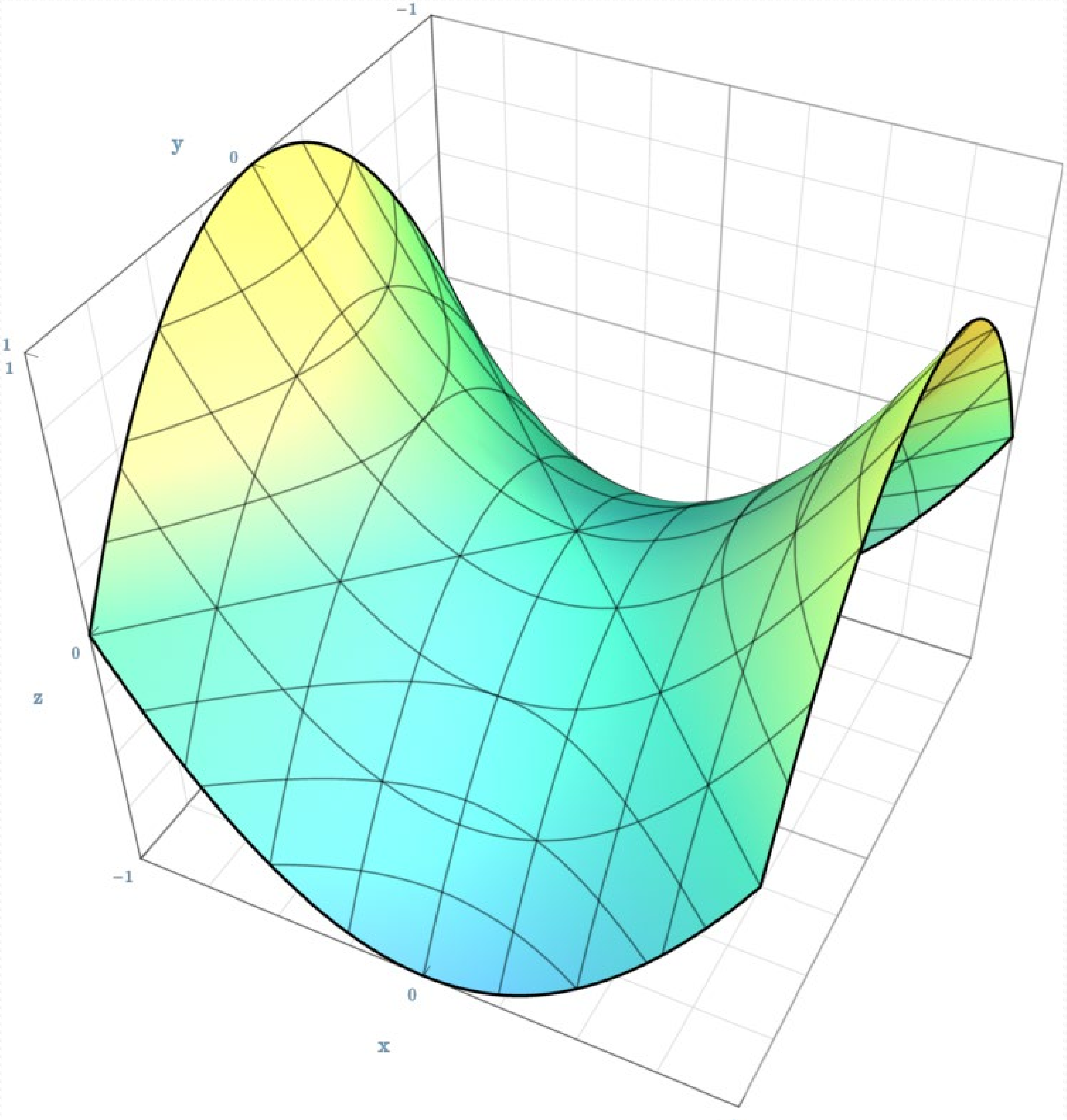


y
x
z
−1
0
1
1
0
−1
0
−1

https://www.khanacademy.org/math/multivariable-calculus

MULTIVARIABLE functions

Functions of Two or More Independent Variables

Usually we study equations of the form $y = f(x)$ where x is the independent variable and y is the dependent variable.

An equation of the form $z = f(x, y)$ describes a **function of two independent variables** if for each ordered pair (x, y), there is only one z determined. The variables x and y are *independent variables* and z is a *dependent variable.*

An equation of the form $w = f(x, y, z)$ describes a **function of three independent variables** if for each ordered triple (x, y, z), there is only one w determined.

Calculus

Functions of two variables:

To each point (x, y) of a certain part of the $x-y$ plane, $x \in R$, $y \in R$ or $(x,y) \in R \times R = R^2$, there corresponds a real value z according to some rule $f(x,y)$, then $f(x,y)$ is called a real valued function of two variables x, y and is written as

$$z = f(x, y), x \in R, y \in R,$$

In general, a real valued function of n variables is defined as

$$z = f(x_1, x_2, ..., x_n), \quad (x_1, x_2, ..., x_n) \in R^n, z \in R$$

Where $x_1, x_2, ..., x_n$ are the n independent variables and z is the dependent variable.

Domain of function: The set of points (x, y) in the $x-y$ plane for which $f(x, y)$ is defined is called the domain of the function and is denoted by D.

17.1 Functions of Several Variables

- A function can involve 2 or more variables, e.g.

$$z = f(x, y) = \frac{2}{x^2 - y^2}$$

Example 1 – Functions of Two Variables

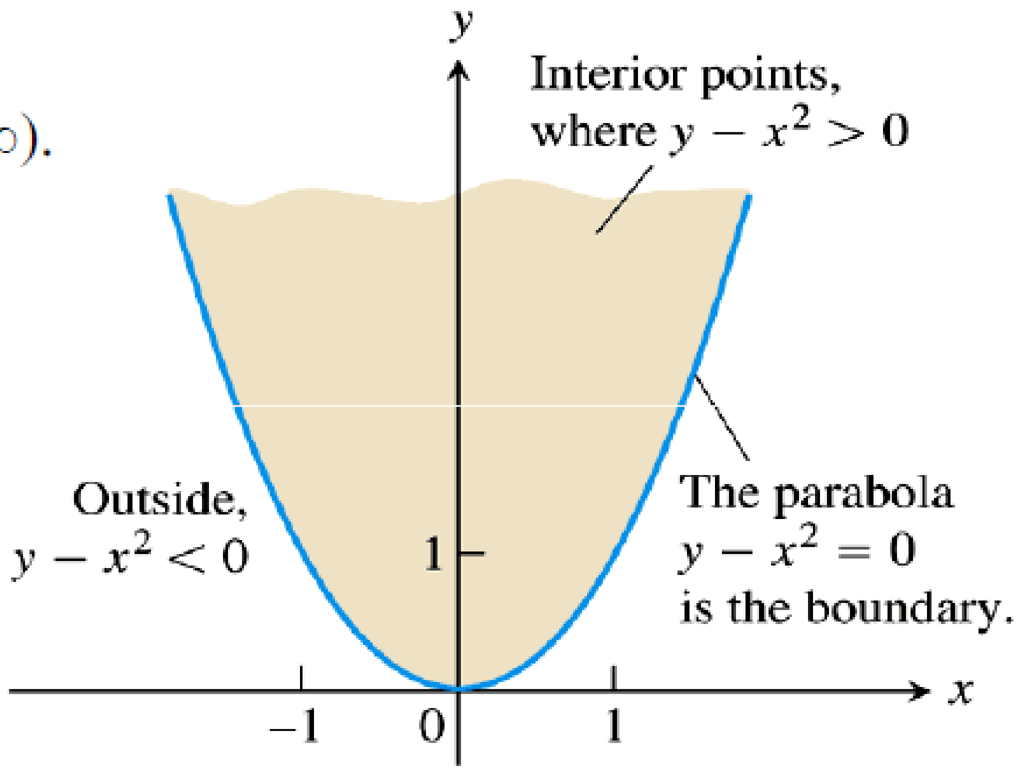
a. $f(x, y) = \dfrac{x+3}{y-2}$ is a function of two variables. Because the denominator is zero when $y = 2$, the domain of f is all (x, y) such that $y \neq 2$.

b. $h(x, y) = 4x$ defines h as a function of x and y. The domain is all ordered pairs of real numbers.

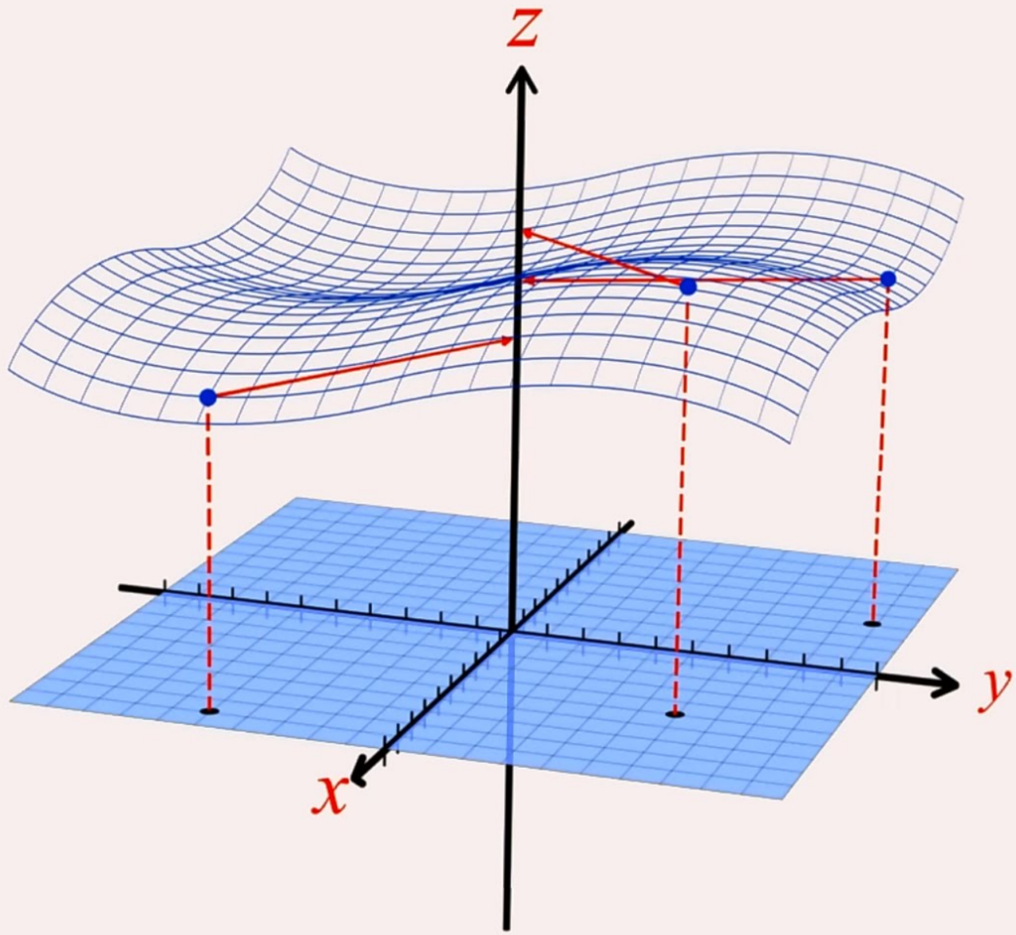
c. $z^2 = x^2 + y^2$ does not define z as a function of x and y.

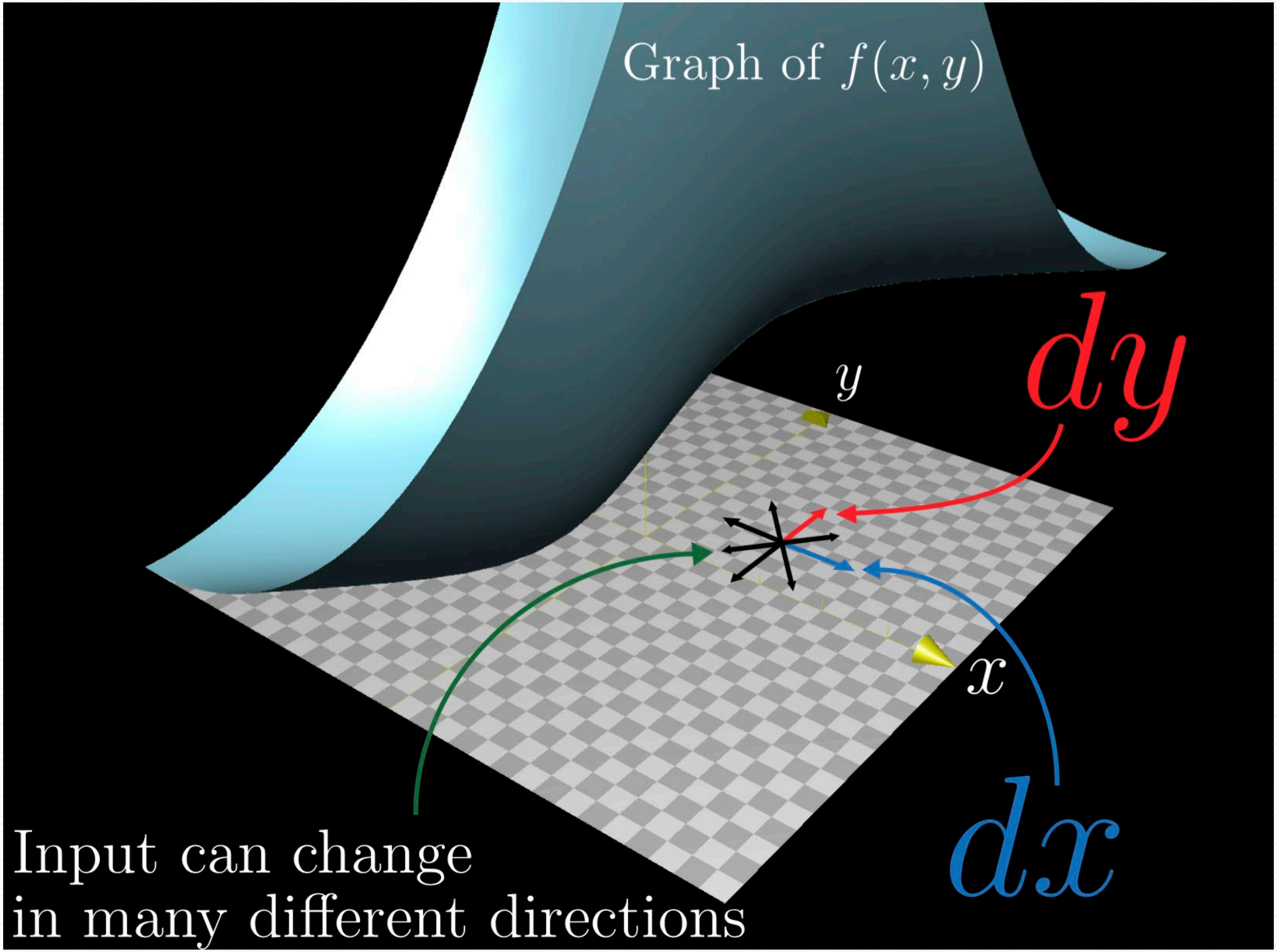
Domain and Range of $f(x, y)$

The range of $f(x, y)$ is $[0, \infty)$.



The domain of $f(x, y) = \sqrt{y - x^2}$ consists of the shaded region and its bounding parabola $y = x^2$.


z
y
x



Graph of $f(x, y)$
dy
y
x
dx
Input can change
in many different directions

PARTIAL DIFFERENTIATION

The process of differentiating a function of several variables with respect to one of its variables while keeping the other variable(s) fixed is called partial differentiation, and the resulting derivative is a partial derivative of the function.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(f(x,y)\right) = z_x = \frac{\partial z}{\partial x} = D_x f$$

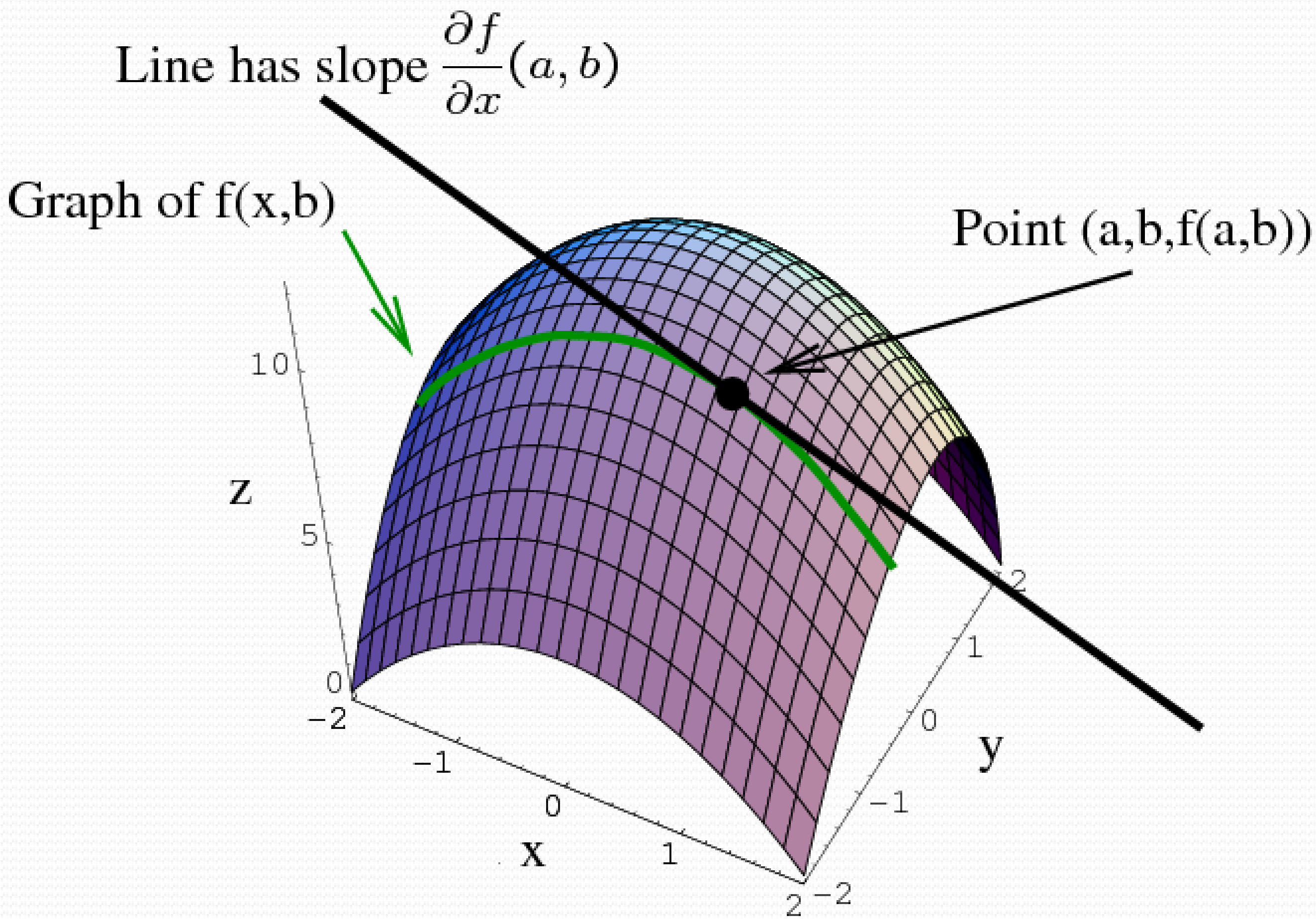
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(f(x,y)\right) = z_y = \frac{\partial z}{\partial y} = D_y f$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

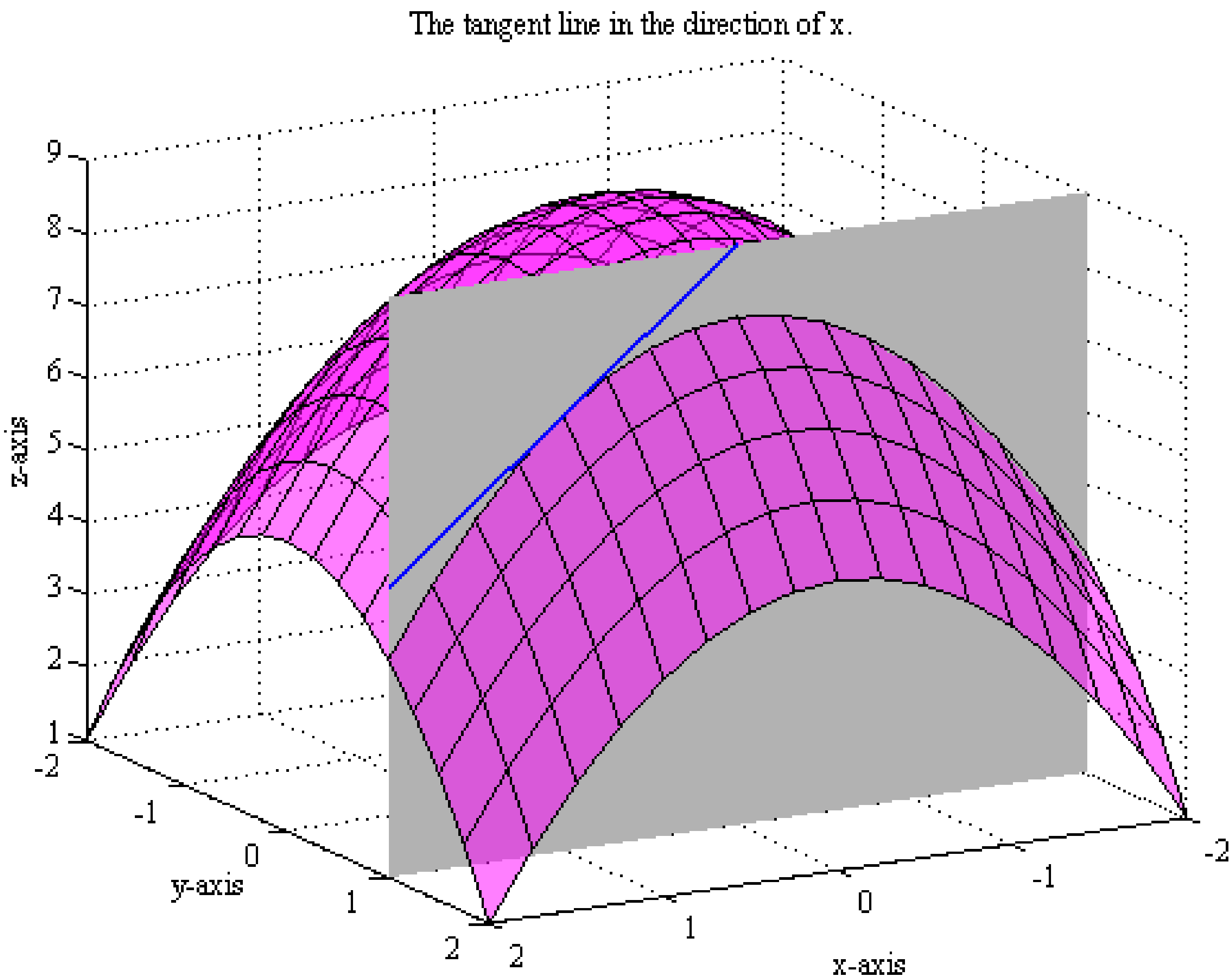
$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

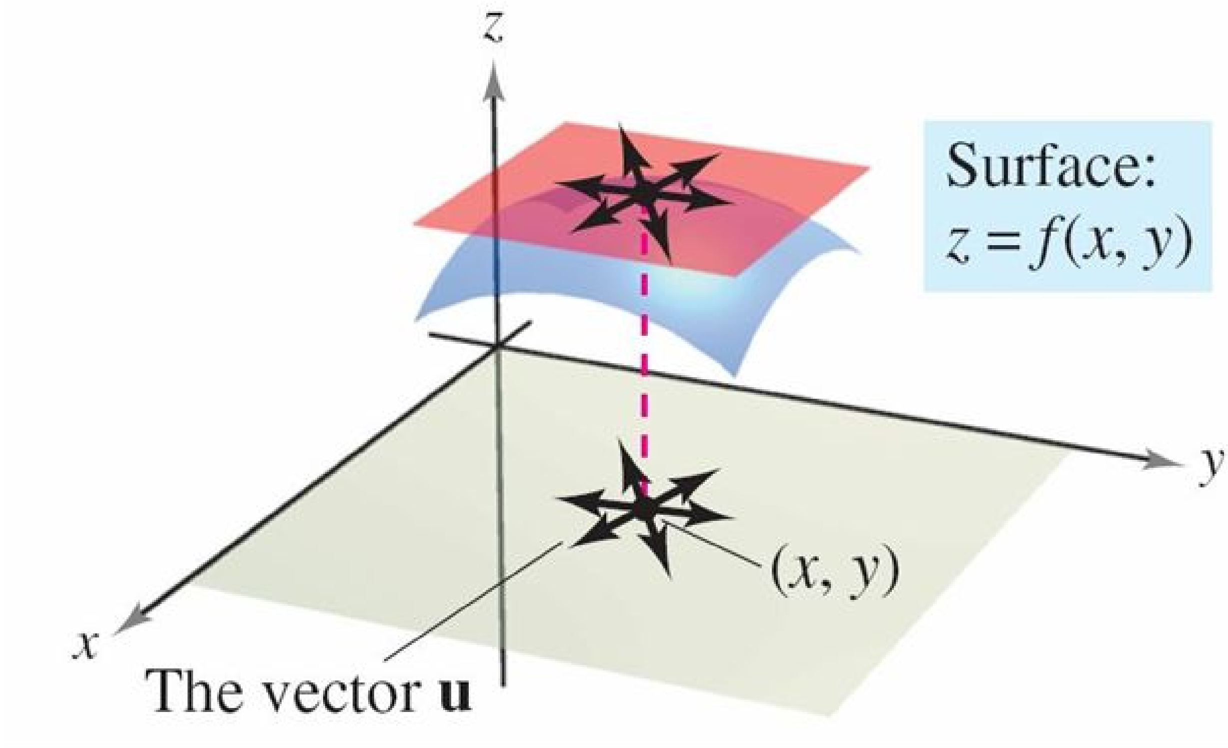


Line has slope $\frac{\partial f}{\partial x}(a,b)$
Graph of f(x,b)
Point (a,b,f(a,b))
z
10
5
0
x
−2
−1
0
1
2
y
−2
−1
0
1
2


The tangent line in the direction of x.
z-axis
9
8
7
6
5
4
3
2
1
y-axis
-2
-1
0
1
2
x-axis
2
1
0
-1
-2

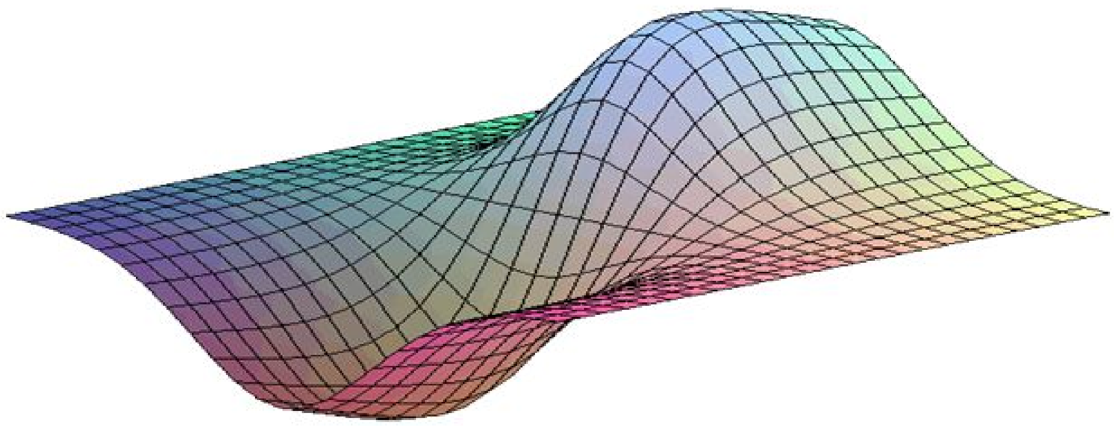
Directional Derivative

There are infinitely many directional derivatives of a surface at a given point—one for each direction specified by **u**, as shown.



Directional Derivatives and Gradients

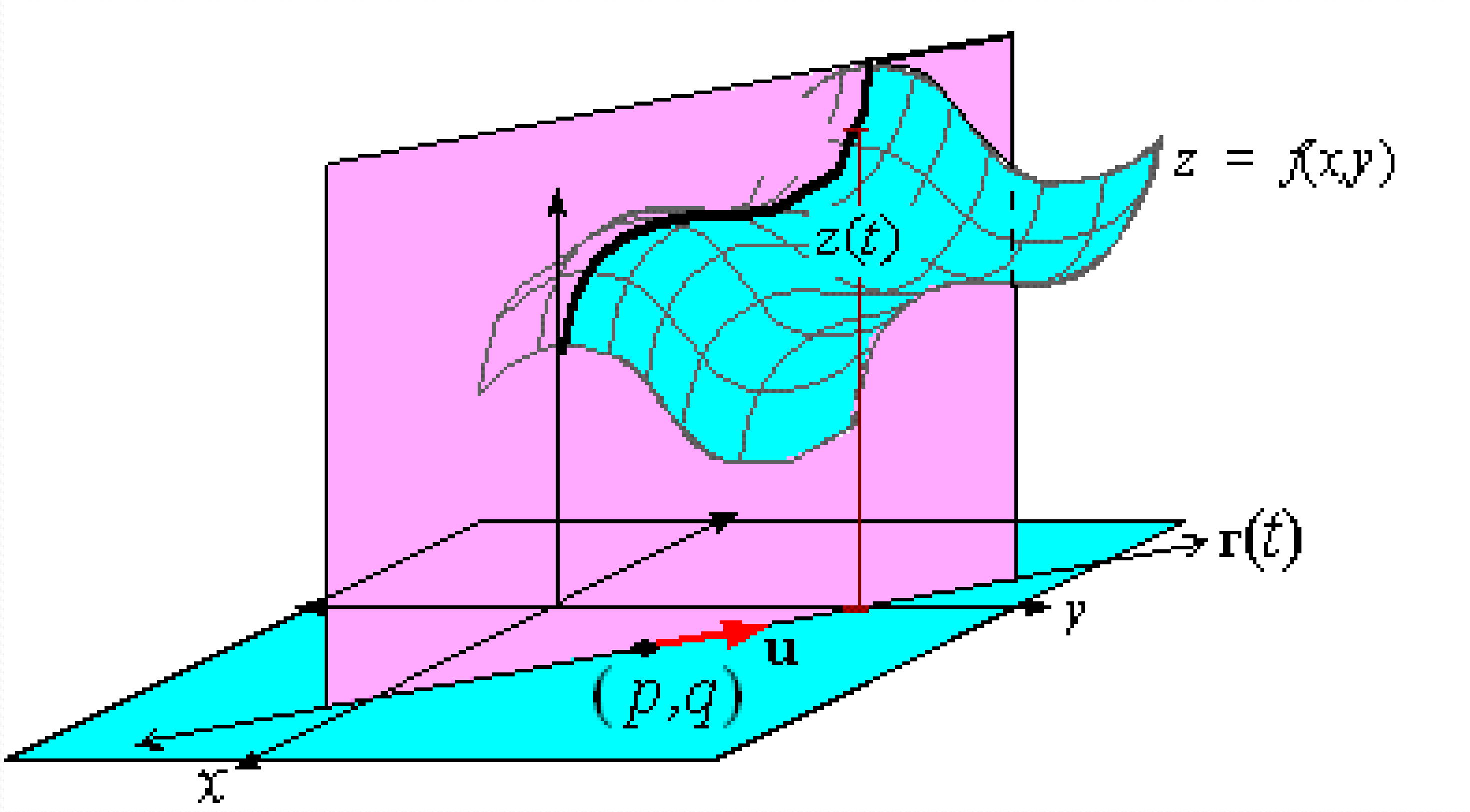
To determine the slope at a point on a surface, we define new type of derivative called a **Directional Derivative**. And to determine in which direction at that point on that surface the slope is maximum, we introduce **Gradient**.



$$f(x,y)$$

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}^T = \textit{gradient}$$

$$= \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j \quad \text{(in two dimensions)}$$


z = f(x,y)
z(t)
r(t)
y
u
(p,q)
x

Directional Derivative and Gradient

- We can now rewrite the directional derivative as

9

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

which expresses the directional derivative in the direction of **u** as the scalar projection of the gradient vector onto **u**.

*EXAMPLE 2

- **Find the directional derivative of the function f (x, y) = x^2y^3 – 4y at the point (2, –1) in the direction of the vector v = 2 i + 5 j.**

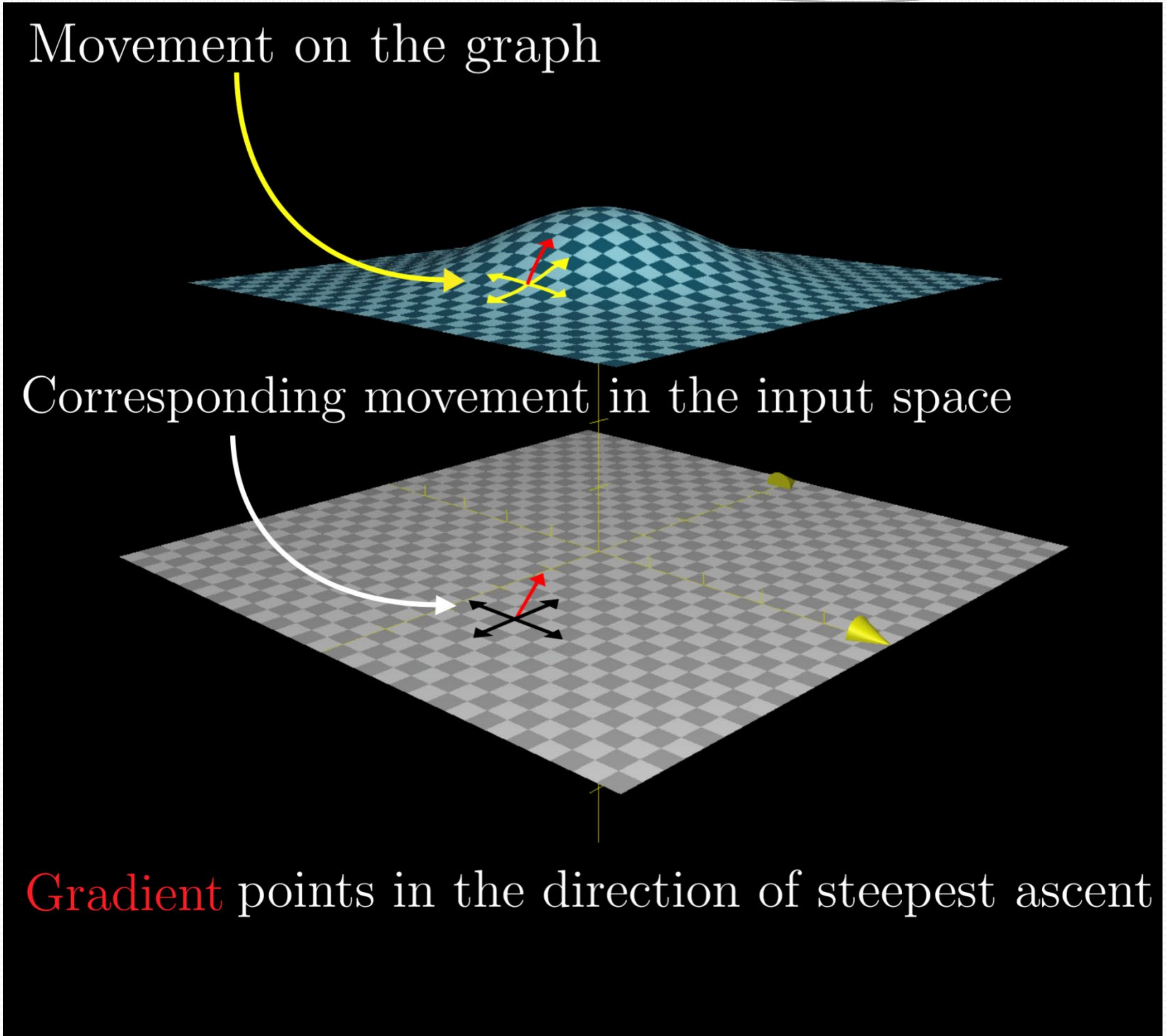
<u>Solution:</u> *by definition,* $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$$\nabla f = i(2xy^3) + j(3x^2y^2 - 4)$$

at (2,-1) $= -4i + 8j$

Directional derivative in the direction of the vector $2\,\mathbf{i} + 5\,\mathbf{j}$

$$= \nabla f\,.\,\frac{a}{|a|} = (-4i + 8j).\,\frac{3\,\mathbf{i} + 4\,\mathbf{j}}{|\sqrt{9+16}|} = \frac{\mathbf{32}}{\mathbf{5}} \quad \textbf{ans}$$


Movement on the graph
Corresponding movement in the input space
Gradient points in the direction of steepest ascent