15

Vector Calculus

15.1 Vector Fields

This chapter is concerned with applying calculus in the context of **vector fields**. A two-dimensional vector field is a function f that maps each point (x,y) in $\mathbb{R}^2$ to a two-dimensional vector $\langle u,v\rangle$, and similarly a three-dimensional vector field maps (x,y,z) to $\langle u,v,w\rangle$. Since a vector has no position, we typically indicate a vector field in graphical form by placing the vector $f(x,y)$ with its tail at (x,y). Figure 15.1 shows a representation of the vector field $f(x,y) = \langle x/\sqrt{x^2+y^2+4}, -y/\sqrt{x^2+y^2+4}\rangle$. For such a graph to be readable, the vectors must be fairly short, which is accomplished by using a different scale for the vectors than for the axes. Such graphs are thus useful for understanding the sizes of the vectors relative to each other but not their absolute size.

Vector fields have many important applications, as they can be used to represent many physical quantities: the vector at a point may represent the strength of some force (gravity, electricity, magnetism) or a velocity (wind speed or the velocity of some other fluid).

We have already seen a particularly important kind of vector field—the gradient. Given a function $f(x,y)$, recall that the gradient is $\langle f_x(x,y), f_y(x,y)\rangle$, a vector that depends on (is a function of) x and y. We usually picture the gradient vector with its tail at (x,y), pointing in the direction of maximum increase. Vector fields that are gradients have some particularly nice properties, as we will see. An important example is

$$\mathbf{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle,$$

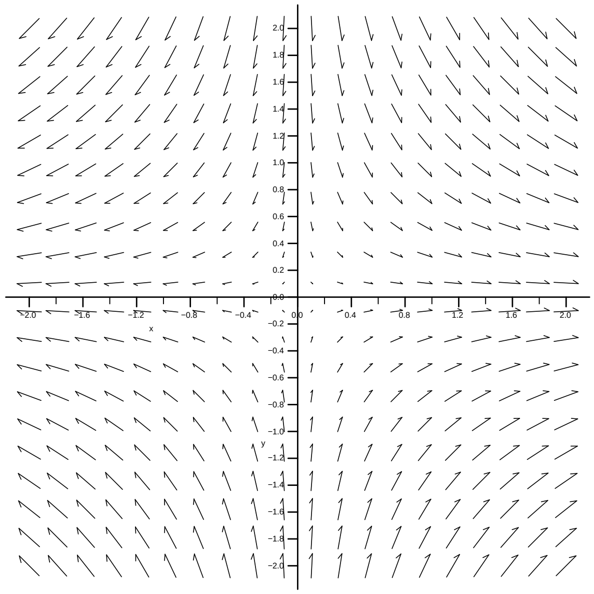

Figure 15.1 A vector field.

which points from the point (x, y, z) toward the origin and has length

$$\frac{\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{(\sqrt{x^2+y^2+z^2})^2},$$

which is the reciprocal of the square of the distance from (x, y, z) to the origin—in other words, $\mathbf{F}$ is an "inverse square law". The vector $\mathbf{F}$ is a gradient:

$$\mathbf{F} = \nabla \frac{1}{\sqrt{x^2+y^2+z^2}}, \tag{15.1}$$

which turns out to be extremely useful.

Exercises

Sketch the vector fields; check your work with Maple's `fieldplot` command.

1. $\langle x, y\rangle$

2. $\langle -x, -y\rangle$

3. $\langle x, -y\rangle$

4. $\langle \sin x, \cos y\rangle$

5. $\langle y, 1/x\rangle$

6. $\langle x+1, x+3\rangle$

7. Verify equation 15.1.

15.2 Line Integrals

We have so far integrated "over" intervals, areas, and volumes with single, double, and triple integrals. We now investigate integration over or "along" a curve—"line integrals" are really "curve integrals".

As with other integrals, a geometric example may be easiest to understand. Consider the function $f = x + y$ and the parabola $y = x^2$ in the x-y plane, for $0 \le x \le 2$. Imagine that we extend the parabola up to the surface f, to form a curved wall or curtain, as in figure 15.2. What is the area of the surface thus formed? We already know one way to compute surface area, but here we take a different approach that is more useful for the problems to come.

As usual, we start by thinking about how to approximate the area. We pick some points along the part of the parabola we're interested in, and connect adjacent points by straight lines; when the points are close together, the length of each line segment will be close to the length along the parabola. Using each line segment as the base of a rectangle, we choose the height to be the height of the surface f above the line segment. If we add up the areas of these rectangles, we get an approximation to the desired area, and in the limit this sum turns into an integral.

Typically the curve is in vector form, or can easily be put in vector form; in this example we have $\mathbf{v}(t) = \langle t, t^2\rangle$. Then as we have seen in section 12.3 on arc length, the length of one of the straight line segments in the approximation is approximately $ds = |\mathbf{v}'|\,dt = \sqrt{1+4t^2}\,dt$, so the integral is

$$\int_0^2 f(t,t^2)\sqrt{1+4t^2}\,dt = \int_0^2 (t+t^2)\sqrt{1+4t^2}\,dt = \frac{167}{48}\sqrt{17} - \frac{1}{12} - \frac{1}{64}\ln(4+\sqrt{17}).$$

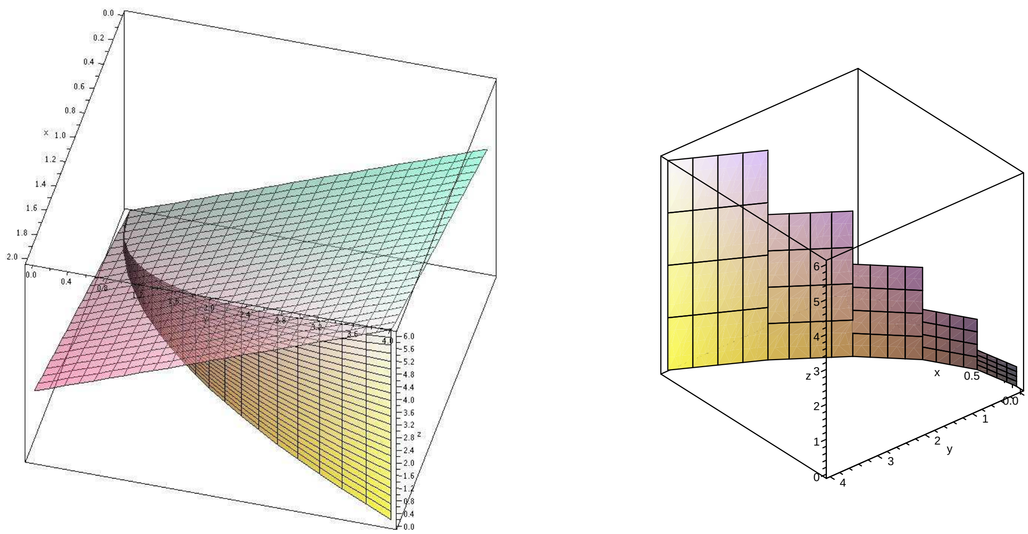

Figure 15.2 Approximating the area under a curve. (JA)

This integral of a function along a curve C is often written in abbreviated form as

$$\int_C f(x, y)\, ds.$$

EXAMPLE 15.1 Compute $\int_C ye^x\, ds$ where C is the line segment from $(1, 2)$ to $(4, 7)$.

We write the line segment as a vector function: $\mathbf{v} = \langle 1, 2\rangle + t\langle 3, 5\rangle$, $0 \le t \le 1$, or in parametric form $x = 1 + 3t$, $y = 2 + 5t$. Then

$$\int_C ye^x\, ds = \int_0^1 (2 + 5t)e^{1+3t}\sqrt{3^2 + 5^2}\, dt = \frac{16}{9}\sqrt{34}e^4 - \frac{1}{9}\sqrt{34}\, e.$$

□

All of these ideas extend to three dimensions in the obvious way.

EXAMPLE 15.2 Compute $\int_C x^2 z\, ds$ where C is the line segment from $(0, 6, -1)$ to $(4, 1, 5)$.

We write the line segment as a vector function: $\mathbf{v} = \langle 0, 6, -1\rangle + t\langle 4, -5, 6\rangle$, $0 \le t \le 1$, or in parametric form $x = 4t$, $y = 6 - 5t$, $z = -1 + 6t$. Then

$$\int_C x^2 z\,ds = \int_0^1 (4t)^2(-1+6t)\sqrt{16+25+36}\,dt = 16\sqrt{77}\int_0^1 -t^2 + 6t^3\,dt = \frac{56}{3}\sqrt{77}.$$

□

Now we turn to a perhaps more interesting example. Recall that in the simplest case, the work done by a force on an object is equal to the magnitude of the force times the distance the object moves; this assumes that the force is constant and in the direction of motion. We have already dealt with examples in which the force is not constant; now we are prepared to examine what happens when the force is not parallel to the direction of motion.

We have already examined the idea of components of force, in example 11.4: the component of a force $\mathbf{F}$ in the direction of a vector $\mathbf{v}$ is

$$\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v},$$

the projection of $\mathbf{F}$ onto $\mathbf{v}$. The length of this vector, that is, the magnitude of the force in the direction of $\mathbf{v}$, is

$$\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|},$$

the scalar projection of $\mathbf{F}$ onto $\mathbf{v}$. If an object moves subject to this (constant) force, in the direction of $\mathbf{v}$, over a distance equal to the length of $\mathbf{v}$, the work done is

$$\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|}|\mathbf{v}| = \mathbf{F}\cdot\mathbf{v}.$$

Thus, work in the vector setting is still "force times distance", except that "times" means "dot product".

If the force varies from point to point, it is represented by a vector field $\mathbf{F}$; the displacement vector $\mathbf{v}$ may also change, as an object may follow a curving path in two or three dimensions. Suppose that the path of an object is given by a vector function $\mathbf{r}(t)$; at any point along the path, the (small) tangent vector $\mathbf{r}'\,\Delta t$ gives an approximation to its motion over a short time Δt, so the work done during that time is approximately $\mathbf{F}\cdot\mathbf{r}'\,\Delta t$; the total work over some time period is then

$$\int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{r}'\,dt.$$

It is useful to rewrite this in various ways at different times. We start with

$$\int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{r}'\,dt = \int_C \mathbf{F}\cdot\,d\mathbf{r},$$

abbreviating $\mathbf{r}'\,dt$ by $d\mathbf{r}$. Or we can write

$$\int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{r}'\,dt = \int_{t_0}^{t_1} \mathbf{F}\cdot\frac{\mathbf{r}'}{|\mathbf{r}'|}|\mathbf{r}'|\,dt = \int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{T}\,|\mathbf{r}'|\,dt = \int_C \mathbf{F}\cdot\mathbf{T}\,ds,$$

using the unit tangent vector $\mathbf{T}$, abbreviating $|\mathbf{r}'|\,dt$ as ds, and indicating the path of the object by C. In other words, work is computed using a particular line integral of the form we have considered. Alternately, we sometimes write

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{r}'\,dt &= \int_C \langle f,g,h\rangle\cdot\langle x',y',z'\rangle\,dt = \int_C f\frac{dx}{dt}+g\frac{dy}{dt}+h\frac{dz}{dt}\,dt\\ &= \int_C f\,dx+g\,dy+h\,dz = \int_C f\,dx+\int_C g\,dy+\int_C h\,dz,\end{aligned}$$

and similarly for two dimensions, leaving out references to z.

EXAMPLE 15.3 Suppose an object moves from $(-1,1)$ to $(2,4)$ along the path $\mathbf{r}(t) = \langle t,t^2\rangle$, subject to the force $\mathbf{F}=\langle x\sin y, y\rangle$. Find the work done.

We can write the force in terms of t as $\langle t\sin(t^2),t^2\rangle$, and compute $\mathbf{r}'(t)=\langle 1,2t\rangle$, and then the work is

$$\int_{-1}^{2} \langle t\sin(t^2),t^2\rangle\cdot\langle 1,2t\rangle\,dt = \int_{-1}^{2} t\sin(t^2)+2t^3\,dt = \frac{15}{2}+\frac{\cos(1)-\cos(4)}{2}.$$

Alternately, we might write

$$\int_C x\sin y\,dx+\int_C y\,dy = \int_{-1}^{2} x\sin(x^2)\,dx+\int_1^4 y\,dy = -\frac{\cos(4)}{2}+\frac{\cos(1)}{2}+\frac{16}{2}-\frac{1}{2}$$

getting the same answer. □

Exercises

1. Compute $\int_C xy^2\,ds$ along the line segment from $(1,2,0)$ to $(2,1,3)$. ⇒

2. Compute $\int_C \sin x\,ds$ along the line segment from $(-1,2,1)$ to $(1,2,5)$. ⇒

3. Compute $\int_C z\cos(xy)\,ds$ along the line segment from $(1,0,1)$ to $(2,2,3)$. ⇒

4. Compute $\int_C \sin x\,dx + \cos y\,dy$ along the top half of the unit circle, from $(1,0)$ to $(-1,0)$. ⇒

5. Compute $\int_C xe^y\,dx + x^2y\,dy$ along the line segment $y=3$, $0\le x\le 2$. ⇒

6. Compute $\int_C xe^y\,dx + x^2y\,dy$ along the line segment $x=4$, $0\le y\le 4$. ⇒

7. Compute $\int_C xe^y\,dx + x^2y\,dy$ along the curve $x=3t$, $y=t^2$, $0\le t\le 1$. ⇒

8. Compute $\int_C xe^y\,dx + x^2y\,dy$ along the curve $\langle e^t, e^t\rangle$, $-1\le t\le 1$. ⇒

9. Compute $\int_C \langle \cos x, \sin y\rangle\cdot d\mathbf{r}$ along the curve $\langle t,t\rangle$, $0\le t\le 1$. ⇒

10. Compute $\int_C \langle 1/xy, 1/(x+y)\rangle\cdot d\mathbf{r}$ along the path from $(1,1)$ to $(3,1)$ to $(3,6)$ using straight line segments. ⇒

11. Compute $\int_C \langle 1/xy, 1/(x+y)\rangle\cdot d\mathbf{r}$ along the curve $\langle 2t, 5t\rangle$, $1\le t\le 4$. ⇒

12. Compute $\int_C \langle 1/xy, 1/(x+y)\rangle\cdot d\mathbf{r}$ along the curve $\langle t, t^2\rangle$, $1\le t\le 4$. ⇒

13. Compute $\int_C yz\,dx + xz\,dy + xy\,dz$ along the curve $\langle t, t^2, t^3\rangle$, $0\le t\le 1$. ⇒

14. Compute $\int_C yz\,dx + xz\,dy + xy\,dz$ along the curve $\langle \cos t, \sin t, \tan t\rangle$, $0\le t\le \pi$. ⇒

15. An object moves from $(1,1)$ to $(4,8)$ along the path $\mathbf{r}(t)=\langle t^2,t^3\rangle$, subject to the force $\mathbf{F}=\langle x^2,\sin y\rangle$. Find the work done. ⇒

16. An object moves along the line segment from $(1,1)$ to $(2,5)$, subject to the force $\mathbf{F}=\langle x/(x^2+y^2), y/(x^2+y^2)\rangle$. Find the work done. ⇒

17. An object moves along the parabola $\mathbf{r}(t)=\langle t,t^2\rangle$, $0\le t\le 1$, subject to the force $\mathbf{F}=\langle 1/(y+1), -1/(x+1)\rangle$. Find the work done. ⇒

18. An object moves along the line segment from $(0,0,0)$ to $(3,6,10)$, subject to the force $\mathbf{F}=\langle x^2,y^2,z^2\rangle$. Find the work done. ⇒

19. An object moves along the curve $\mathbf{r}(t)=\langle\sqrt{t}, 1/\sqrt{t}, t\rangle$ $1\le t\le 4$, subject to the force $\mathbf{F}=\langle y,z,x\rangle$. Find the work done. ⇒

20. An object moves from $(1,1,1)$ to $(2,4,8)$ along the path $\mathbf{r}(t)=\langle t,t^2,t^3\rangle$, subject to the force $\mathbf{F}=\langle \sin x, \sin y, \sin z\rangle$. Find the work done. ⇒

21. An object moves from $(1,0,0)$ to $(-1,0,\pi)$ along the path $\mathbf{r}(t)=\langle\cos t,\sin t,t\rangle$, subject to the force $\mathbf{F}=\langle y^2,y^2,xz\rangle$. Find the work done. ⇒

15.3 The Fundamental Theorem of Line Integrals

One way to write the Fundamental Theorem of Calculus (7.3) is:

$$\int_a^b f'(x)\,dx = f(b)-f(a).$$

That is, to compute the integral of a derivative f' we need only compute the values of f at the endpoints. Something similar is true for line integrals of a certain form.

THEOREM 15.4 Fundamental Theorem of Line Integrals Suppose a curve C is given by the vector function $\mathbf{r}(t)$, with $\mathbf{a}=\mathbf{r}(a)$ and $\mathbf{b}=\mathbf{r}(b)$. Then

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{b})-f(\mathbf{a}),$$

provided that $\mathbf{r}$ is sufficiently nice.

Proof.

We write $\mathbf{r}=\langle x(t),y(t),z(t)\rangle$, so that $\mathbf{r}'=\langle x'(t),y'(t),z'(t)\rangle$. Also, we know that $\nabla f=\langle f_x,f_y,f_z\rangle$. Then

$$\int_C \nabla f\cdot d\mathbf{r} = \int_a^b \langle f_x,f_y,f_z\rangle\cdot\langle x'(t),y'(t),z'(t)\rangle\,dt = \int_a^b f_xx'+f_yy'+f_zz'\,dt.$$

By the chain rule (see section 13.4) $f_xx'+f_yy'+f_zz'=df/dt$, where f in this context means $f(x(t),y(t),z(t))$, a function of t. In other words, all we have is

$$\int_a^b f'(t)\,dt = f(b)-f(a).$$

In this context, $f(a)=f(x(a),y(a),z(a))$. Since $\mathbf{a}=\mathbf{r}(a)=\langle x(a),y(a),z(a)\rangle$, we can write $f(a)=f(\mathbf{a})$—this is a bit of a cheat, since we are simultaneously using f to mean $f(t)$ and $f(x,y,z)$, and since $f(x(a),y(a),z(a))$ is not technically the same as $f(\langle x(a),y(a),z(a)\rangle)$, but the concepts are clear and the different uses are compatible. Doing the same for b, we get

$$\int_C \nabla f\cdot d\mathbf{r} = \int_a^b f'(t)\,dt = f(b)-f(a) = f(\mathbf{b})-f(\mathbf{a}).$$

■

This theorem, like the Fundamental Theorem of Calculus, says roughly that if we integrate a “derivative-like function” (f' or ∇f) the result depends only on the values of the original function (f) at the endpoints.

If a vector field $\mathbf{F}$ is the gradient of a function, $\mathbf{F} = \nabla f$, we say that $\mathbf{F}$ is a **conservative vector field**. If $\mathbf{F}$ is a conservative force field, then the integral for work, $\int_C \mathbf{F} \cdot d\mathbf{r}$, is in the form required by the Fundamental Theorem of Line Integrals. This means that in a conservative force field, the amount of work required to move an object from point $\mathbf{a}$ to point $\mathbf{b}$ depends only on those points, not on the path taken between them.

EXAMPLE 15.5 An object moves in the force field

$$\mathbf{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle,$$

along the curve $\mathbf{r} = \langle 1+t, t^3, t\cos(\pi t)\rangle$ as t ranges from 0 to 1. Find the work done by the force on the object.

The straightforward way to do this involves substituting the components of $\mathbf{r}$ into $\mathbf{F}$, forming the dot product $\mathbf{F}\cdot\mathbf{r}'$, and then trying to compute the integral, but this integral is extraordinarily messy, perhaps impossible to compute. But since $\mathbf{F} = \nabla(1/\sqrt{x^2+y^2+z^2})$ we need only substitute:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left.\frac{1}{\sqrt{x^2+y^2+z^2}}\right|_{(1,0,0)}^{(2,1,-1)} = \frac{1}{\sqrt{6}} - 1.$$

□

Another immediate consequence of the Fundamental Theorem involves **closed paths**. A path C is closed if it forms a loop, so that traveling over the C curve brings you back to the starting point. If C is a closed path, we can integrate around it starting at any point $\mathbf{a}$; since the starting and ending points are the same,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{a}) - f(\mathbf{a}) = 0.$$

For example, in a gravitational field (an inverse square law field) the amount of work required to move an object around a closed path is zero. Of course, it’s only the *net* amount of work that is zero. It may well take a great deal of work to get from point $\mathbf{a}$ to point **b**, but then the return trip will "produce" work. For example, it takes work to pump water from a lower to a higher elevation, but if you then let gravity pull the water back down, you can recover work by running a water wheel or generator. (In the real world you won't recover all the work because of various losses along the way.)

To make use of the Fundamental Theorem of Line Integrals, we need to be able to spot conservative vector fields $\mathbf{F}$ and to compute f so that $\mathbf{F} = \nabla f$. Suppose that $\mathbf{F} = \langle P, Q\rangle = \nabla f$. Then $P = f_x$ and $Q = f_y$, and provided that f is sufficiently nice, we know from Clairaut's Theorem (13.23) that $P_y = f_{xy} = f_{yx} = Q_x$. If we compute P_y and Q_x and find that they are not equal, then $\mathbf{F}$ is not conservative. If $P_y = Q_x$, then, again provided that $\mathbf{F}$ is sufficiently nice, we can be assured that $\mathbf{F}$ is conservative. Ultimately, what's important is that we be able to find f; as this amounts to finding anti-derivatives, we may not always succeed.

EXAMPLE 15.6 Find f so that $\langle 3 + 2xy, x^2 - 3y^2\rangle = \nabla f$.

First, note that

$$\frac{\partial}{\partial y}(3 + 2xy) = 2x \qquad \text{and} \qquad \frac{\partial}{\partial x}(x^2 - 3y^2) = 2x,$$

so the desired f does exist. This means that $f_x = 3 + 2xy$, so that $f = 3x + x^2y + g(y)$; the first two terms are needed to get $3+2xy$, and the $g(y)$ could be any function of y, as it would disappear upon taking a derivative with respect to x. Likewise, since $f_y = x^2 - 3y^2$, $f = x^2y - y^3 + h(x)$. The question now becomes, is it possible to find $g(y)$ and $h(x)$ so that

$$3x + x^2y + g(y) = x^2y - y^3 + h(x),$$

and of course the answer is yes: $g(y) = -y^3$, $h(x) = 3x$. Thus, $f = 3x + x^2y - y^3$. □

Exercises

1. Find f so that $\nabla f = \langle 2x + y^2, 2y + x^2 \rangle$, or explain why there is no such f. ⇒
2. Find f so that $\nabla f = \langle x^3, -y^4 \rangle$, or explain why there is no such f. ⇒
3. Find f so that $\nabla f = \langle xe^y, ye^x \rangle$, or explain why there is no such f. ⇒
4. Find f so that $\nabla f = \langle y \cos x, y \sin x \rangle$, or explain why there is no such f. ⇒
5. Find f so that $\nabla f = \langle y \cos x, \sin x \rangle$, or explain why there is no such f. ⇒
6. Find f so that $\nabla f = \langle x^2 y^3, xy^4 \rangle$, or explain why there is no such f. ⇒
7. Find f so that $\nabla f = \langle yz, xz, xy \rangle$, or explain why there is no such f. ⇒
8. Let $\mathbf{F} = \langle yz, xz, xy \rangle$. Find the work done by this force field on an object that moves from $(1, 0, 2)$ to $(1, 2, 3)$. ⇒
9. Let $\mathbf{F} = \langle e^y, xe^y + \sin z, y \cos z \rangle$. Find the work done by this force field on an object that moves from $(0, 0, 0)$ to $(1, -1, 3)$. ⇒
10. Let
$$\mathbf{F} = \left\langle \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle.$$
Find the work done by this force field on an object that moves from $(1, 1, 1)$ to $(4, 5, 6)$. ⇒

15.4 Green's Theorem

We now come to the first of three important theorems that extend the Fundamental Theorem of Calculus to higher dimensions. (The Fundamental Theorem of Line Integrals has already done this in one way, but in that case we were still dealing with an essentially one-dimensional integral.) They all share with the Fundamental Theorem the following rather vague description: *To compute a certain sort of integral over a region, we may do a computation on the boundary of the region that involves one fewer integrations.*

Note that this does indeed describe the Fundamental Theorem of Calculus and the Fundamental Theorem of Line Integrals: to compute a single integral over an interval, we do a computation on the boundary (the endpoints) that involves one fewer integrations, namely, no integrations at all.

THEOREM 15.7 Green's Theorem If the vector field $\mathbf{F} = \langle P, Q \rangle$ and the region D are sufficiently nice, and if C is the boundary of D (C is a closed curve), then

$$\iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \int_C P \, dx + Q \, dy,$$

provided the integration on the right is done counter-clockwise around C.

To indicate that an integral $\int_C$ is being done over a closed curve in the counterclockwise direction, we usually write $\oint_C$. We also use the notation ∂D to mean the boundary of D **oriented** in the counterclockwise direction. With this notation, $\oint_C = \int_{\partial D}$.

We already know one case, not particularly interesting, in which this theorem is true: If $\mathbf{F}$ is conservative, we know that the integral $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$, because any integral of a conservative vector field around a closed curve is zero. We also know in this case that $\partial P/\partial y = \partial Q/\partial x$, so the double integral in the theorem is simply the integral of the zero function, namely, 0. So in the case that $\mathbf{F}$ is conservative, the theorem says simply that $0 = 0$.

EXAMPLE 15.8 We illustrate the theorem by computing both sides of

$$\int_{\partial D} x^4\,dx + xy\,dy = \iint_D y - 0\,dA,$$

where D is the triangular region with corners $(0,0)$, $(1,0)$, $(0,1)$.

Starting with the double integral:

$$\iint_D y - 0\,dA = \int_0^1 \int_0^{1-x} y\,dy\,dx = \int_0^1 \frac{(1-x)^2}{2}\,dx = -\frac{(1-x)^3}{6}\bigg|_0^1 = \frac{1}{6}.$$

There is no single formula to describe the boundary of D, so to compute the left side directly we need to compute three separate integrals corresponding to the three sides of the triangle, and each of these integrals we break into two integrals, the "dx" part and the "dy" part. The three sides are described by $y = 0$, $y = 1 - x$, and $x = 0$. The integrals are then

$$\begin{aligned}\int_{\partial D} x^4\,dx + xy\,dy &= \int_0^1 x^4\,dx + \int_0^0 0\,dy + \int_1^0 x^4\,dx + \int_0^1 (1-y)y\,dy + \int_0^0 0\,dx + \int_1^0 0\,dy \\ &= \frac{1}{5} + 0 - \frac{1}{5} + \frac{1}{6} + 0 + 0 = \frac{1}{6}.\end{aligned}$$

Alternately, we could descibe the three sides in vector form as $\langle t, 0\rangle$, $\langle 1-t, t\rangle$, and $\langle 0, 1-t\rangle$. Note that in each case, as t ranges from 0 to 1, we follow the corresponding side in the correct direction. Now

$$\begin{aligned}\int_{\partial D} x^4\,dx + xy\,dy &= \int_0^1 t^4 + t\cdot 0\,dt + \int_0^1 -(1-t)^4 + (1-t)t\,dt + \int_0^1 0 + 0\,dt \\ &= \int_0^1 t^4\,dt + \int_0^1 -(1-t)^4 + (1-t)t\,dt = \frac{1}{6}.\end{aligned}$$

□

In this case, none of the integrations are difficult, but the second approach is somewhat tedious because of the necessity to set up three different integrals. In different circumstances, either of the integrals, the single or the double, might be easier to compute. Sometimes it is worthwhile to turn a single integral into the corresponding double integral, sometimes exactly the opposite approach is best.

Here is a clever use of Green's Theorem: We know that areas can be computed using double integrals, namely,

$$\iint_D 1\,dA$$

computes the area of region D. If we can find P and Q so that $\partial Q/\partial x - \partial P/\partial y = 1$, then the area is also

$$\int_{\partial D} P\,dx + Q\,dy.$$

It is quite easy to do this: $P = 0, Q = x$ works, as do $P = -y, Q = 0$ and $P = -y/2, Q = x/2$.

EXAMPLE 15.9 An ellipse centered at the origin, with its two principal axes aligned with the x and y axes, is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We find the area of the interior of the ellipse via Green's theorem. To do this we need a vector equation for the boundary; one such equation is $\langle a\cos t, b\sin t\rangle$, as t ranges from 0 to 2π. We can easily verify this by substitution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1.$$

Let's consider the three possibilities for P and Q above: Using 0 and x gives

$$\oint_C 0\,dx + x\,dy = \int_0^{2\pi} a\cos(t)b\cos(t)\,dt = \int_0^{2\pi} ab\cos^2(t)\,dt.$$

Using $-y$ and 0 gives

$$\oint_C -y\,dx + 0\,dy = \int_0^{2\pi} -b\sin(t)(-a\sin(t))\,dt = \int_0^{2\pi} ab\sin^2(t)\,dt.$$

Finally, using $-y/2$ and $x/2$ gives

$$\oint_C -\frac{y}{2}\,dx + \frac{x}{2}\,dy = \int_0^{2\pi} -\frac{b\sin(t)}{2}(-a\sin(t))\,dt + \frac{a\cos(t)}{2}(b\cos(t))\,dt$$
$$= \int_0^{2\pi} \frac{ab\sin^2 t}{2} + \frac{ab\cos^2 t}{2}\,dt = \int_0^{2\pi} \frac{ab}{2}\,dt = \pi ab.$$

The first two integrals are not particularly difficult, but the third is very easy, though the choice of P and Q seems more complicated. □

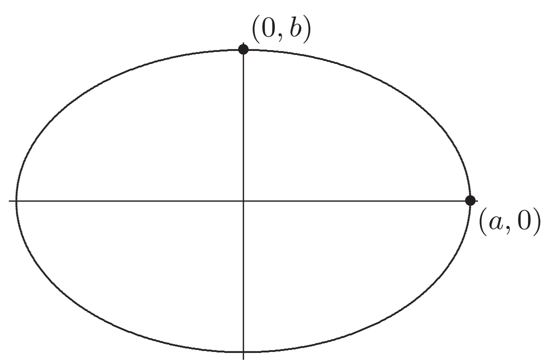

Figure 15.3 A "standard" ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Proof of Green's Theorem.

We cannot here prove Green's Theorem in general, but we can do a special case. We seek to prove that

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$

It is sufficient to show that

$$\oint_C P\,dx = \iint_D -\frac{\partial P}{\partial y}\,dA \qquad \text{and} \qquad \oint_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA,$$

which we can do if we can compute the double integral in both possible ways, that is, using $dA = dy\,dx$ and $dA = dx\,dy$.

For the first equation, we start with

$$\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx = \int_a^b P(x,g_2(x)) - P(x,g_1(x))\,dx.$$

Here we have simply used the ordinary Fundamental Theorem of Calculus, since for the inner integral we are integrating a derivative with respect to y: an antiderivative of $\partial P/\partial y$ with respect to y is simply $P(x,y)$, and then we substitute g_1 and g_2 for y and subtract.

Now we need to manipulate $\oint_C P\,dx$. The boundary of region D consists of 4 parts, given by the equations $y = g_1(x)$, $x = b$, $y = g_2(x)$, and $x = a$. On the portions $x = b$ and $x = a$, $dx = 0\,dt$, so the corresponding integrals are zero. For the other two portions, we use the parametric forms $x = t$, $y = g_1(t)$, $a \le t \le b$, and $x = t$, $y = g_2(t)$, letting t range from b to a, since we are integrating counter-clockwise around the boundary. The resulting integrals give us

$$\begin{aligned}\oint_C P\,dx &= \int_a^b P(t,g_1(t))\,dt + \int_b^a P(t,g_2(t))\,dt = \int_a^b P(t,g_1(t))\,dt - \int_a^b P(t,g_2(t))\,dt\\ &= \int_a^b P(t,g_1(t)) - P(t,g_2(t))\,dt\end{aligned}$$

which is the result of the double integral times -1, as desired.

The equation involving Q is essentially the same, and left as an exercise. ■

Exercises

1. Compute $\int_{\partial D} 2y\,dx + 3x\,dy$, where D is described by $0 \le x, y \le 1$. ⇒

2. Compute $\int_{\partial D} xy\,dx + xy\,dy$, where D is described by $0 \le x, y \le 1$. ⇒

3. Compute $\int_{\partial D} e^{2x+3y}\,dx + e^{xy}\,dy$, where D is described by $-2 \le x \le 2$, $-1 \le y \le 1$. ⇒

4. Compute $\int_{\partial D} y\cos x\,dx + y\sin x\,dy$, where D is described by $0 \le x \le \pi/2$, $1 \le y \le 2$. ⇒

5. Compute $\int_{\partial D} x^2y\,dx + xy^2\,dy$, where D is described by $0 \le x \le 1$, $0 \le y \le x$. ⇒

6. Compute $\int_{\partial D} x\sqrt{y}\,dx + \sqrt{x+y}\,dy$, where D is described by $1 \le x \le 2$, $2x \le y \le 4$. ⇒

7. Compute $\int_{\partial D} (x/y)\,dx + (2+3x)\,dy$, where D is described by $1 \le x \le 2$, $1 \le y \le x^2$. ⇒

8. Compute $\int_{\partial D} \sin y\,dx + \sin x\,dy$, where D is described by $0 \le x \le \pi/2$, $x \le y \le \pi/2$. ⇒

9. Compute $\int_{\partial D} x \ln y \, dx$, where D is described by $1 \le x \le 2$, $e^x \le y \le e^{x^2}$. ⇒

10. Compute $\int_{\partial D} \sqrt{1+x^2} \, dy$, where D is described by $-1 \le x \le 1$, $x^2 \le y \le 1$. ⇒

11. Compute $\int_{\partial D} x^2 y \, dx - xy^2 \, dy$, where D is described by $x^2 + y^2 \le 1$. ⇒

12. Compute $\int_{\partial D} y^3 \, dx + 2x^3 \, dy$, where D is described by $x^2 + y^2 \le 4$. ⇒

13. Finish our proof of Green's Theorem by showing that $\oint_C Q \, dy = \iint\limits_D \frac{\partial Q}{\partial x} \, dA$.

15.5 Divergence and Curl

Divergence and curl are two measurements of vector fields that are very useful in a variety of applications. Both are most easily understood by thinking of the vector field as representing a flow of a liquid or gas; that is, each vector in the vector field should be interpreted as a velocity vector. Roughly speaking, divergence measures the tendency of the fluid to collect or disperse at a point, and curl measures the tendency of the fluid to swirl around the point. Divergence is a scalar, that is, a single number, while curl is itself a vector. The magnitude of the curl measures how much the fluid is swirling, the direction indicates the axis around which it tends to swirl. These ideas are somewhat subtle in practice, and are beyond the scope of this course. You can find additional information on the web, for example at http://www.math.umn.edu/~nykamp/m2374/readings/divcurl/ and in many books including *Div, Grad, Curl, and All That: An Informal Text on Vector Calculus*, by H. M. Schey.

Recall that if f is a function, the gradient of f is given by

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

A useful mnemonic for this (and for the divergence and curl, as it turns out) is to let

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle,$$

that is, we pretend that ∇ is a vector with rather odd looking entries. Recalling that $\langle u, v, w \rangle a = \langle ua, va, wa \rangle$, we can then think of the gradient as

$$\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle,$$

that is, we simply multiply the f into the vector.

The divergence and curl can now be defined in terms of this same odd vector ∇ by using the cross product and dot product. The divergence of a vector field $\mathbf{F} = \langle f, g, h\rangle$ is

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h\rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle.$$

Here are two simple but useful facts about divergence and curl.

THEOREM 15.10 $\nabla \cdot (\nabla \times \mathbf{F}) = 0.$

In words, this says that the divergence of the curl is zero.

THEOREM 15.11 $\nabla \times (\nabla f) = \mathbf{0}.$

That is, the curl of a gradient is the zero vector. Recalling that gradients are conservative vector fields, this says that the curl of a conservative vector field is the zero vector. Under suitable conditions, it is also true that if the curl of $\mathbf{F}$ is $\mathbf{0}$ then $\mathbf{F}$ is conservative.

EXAMPLE 15.12 Let $\mathbf{F} = \langle e^z, 1, xe^z\rangle$. Then $\nabla \times \mathbf{F} = \langle 0, e^z - e^z, 0\rangle = \mathbf{0}$. Thus, $\mathbf{F}$ is conservative, and we can exhibit this directly by finding the corresponding f.

Since $f_x = e^z$, $f = xe^z + g(y, z)$. Since $f_y = 1$, it must be that $g_y = 1$, so $g(y, z) = y + h(z)$. Thus $f = xe^z + y + h(z)$ and

$$xe^z = f_z xe^z + 0 + h'(z),$$

so $h'(z) = 0$, i.e., $h(z) = C$, and $f = xe^z + y + C$. □

We can rewrite Green's Theorem using these new ideas; these rewritten versions in turn are closer to some later theorems we will see.

Suppose we write a two dimensional vector field in the form $\mathbf{F} = \langle P, Q, 0\rangle$, where P and Q are functions of x and y. Then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \langle 0, 0, Q_x - P_y\rangle,$$

and so $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \langle 0, 0, Q_x - P_y \rangle \cdot \langle 0, 0, 1 \rangle = Q_x - P_y$. So Green's Theorem says

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA. \tag{15.2}$$

Roughly speaking, the right-most integral adds up the curl (tendency to swirl) at each point in the region; the left-most integral adds up the tangential components of the vector field around the entire boundary. Green's Theorem says these are equal, or roughly, that the sum of the "microscopic" swirls over the region is the same as the "macroscopic" swirl around the boundary.

Next, suppose that the boundary ∂D has a vector form $\mathbf{r}(t)$, so that $\mathbf{r}'(t)$ is tangent to the boundary, and $\mathbf{T} = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ is the usual unit tangent vector. Writing $\mathbf{r} = \langle x(t), y(t) \rangle$ we get

$$\mathbf{T} = \frac{\langle x', y' \rangle}{|\mathbf{r}'(t)|}$$

and then

$$\mathbf{N} = \frac{\langle y', -x' \rangle}{|\mathbf{r}'(t)|}$$

is a unit vector perpendicular to $\mathbf{T}$, that is, a unit normal to the boundary. Now

$$\begin{aligned}\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\,ds &= \int_{\partial D} \langle P, Q \rangle \cdot \frac{\langle y', -x' \rangle}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)| dt = \int_{\partial D} Py'\,dt - Qx'\,dt \\ &= \int_{\partial D} P\,dy - Q\,dx = \int_{\partial D} -Q\,dx + P\,dy.\end{aligned}$$

So far, we've just rewritten the original integral using alternate notation. The last integral looks just like the left side of Green's Theorem (15.7) except that P and Q have traded places and Q has acquired a negative sign. Then applying Green's Theorem we get

$$\int_{\partial D} -Q\,dx + P\,dy = \iint_D P_x + Q_y\,dA = \iint_D \nabla \cdot \mathbf{F}\,dA.$$

Summarizing the long string of equalities,

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\,ds = \iint_D \nabla \cdot \mathbf{F}\,dA. \tag{15.3}$$

Roughly speaking, the first integral adds up the flow across the boundary of the region, from inside to out, and the second sums the divergence (tendency to spread) at each point in the interior. The theorem roughly says that the sum of the "microscopic" spreads is the same as the total spread across the boundary and out of the region.

Exercises

1. Let $\mathbf{F} = \langle xy, -xy\rangle$ and let D be given by $0 \le x \le 1$, $0 \le y \le 1$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

2. Let $\mathbf{F} = \langle ax^2, by^2\rangle$ and let D be given by $0 \le x \le 1$, $0 \le y \le 1$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

3. Let $\mathbf{F} = \langle ay^2, bx^2\rangle$ and let D be given by $0 \le x \le 1$, $0 \le y \le x$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

4. Let $\mathbf{F} = \langle \sin x\cos y, \cos x\sin y\rangle$ and let D be given by $0 \le x \le \pi/2$, $0 \le y \le x$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

5. Let $\mathbf{F} = \langle y, -x\rangle$ and let D be given by $x^2 + y^2 \le 1$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

6. Let $\mathbf{F} = \langle x, y\rangle$ and let D be given by $x^2 + y^2 \le 1$. Compute $\int_{\partial D} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds$. ⇒

15.6 Vector Equations of Surfaces

We have dealt extensively with vector equations for curves, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. A similar technique can be used to represent surfaces in a way that is more general than the equations for surfaces we have used so far. Recall that when we use $\mathbf{r}(t)$ to represent a curve, we imagine the vector $\mathbf{r}(t)$ with its tail at the origin, and then we follow the head of the arrow as t changes. The vector "draws" the curve through space as t varies.

Suppose we instead have a vector function of two variables,

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle.$$

As both u and v vary, we again imagine the vector $\mathbf{r}(u, v)$ with its tail at the origin, and its head sweeps out a surface in space. A useful analogy is the technology of CRT video screens, in which an electron gun fires electrons in the direction of the screen. The gun's direction sweeps horizontally and vertically to "paint" the screen with the desired image. In practice, the gun moves horizontally through an entire line, then moves vertically to the next line and repeats the operation. In the same way, it can be useful to imagine fixing a value of v and letting $\mathbf{r}(u, v)$ sweep out a curve as u changes. Then v can change a bit, and $\mathbf{r}(u, v)$ sweeps out a new curve very close to the first. Put enough of these curves together and they form a surface.

EXAMPLE 15.13 Consider the function $\mathbf{r}(u, v) = \langle v \cos u, v \sin u, v\rangle$. For a fixed value of v, as u varies from 0 to 2π, this traces a circle in space at height v above the x-y plane and with radius v. Put lots and lots of these together,and they form a cone, as in figure 15.4. □

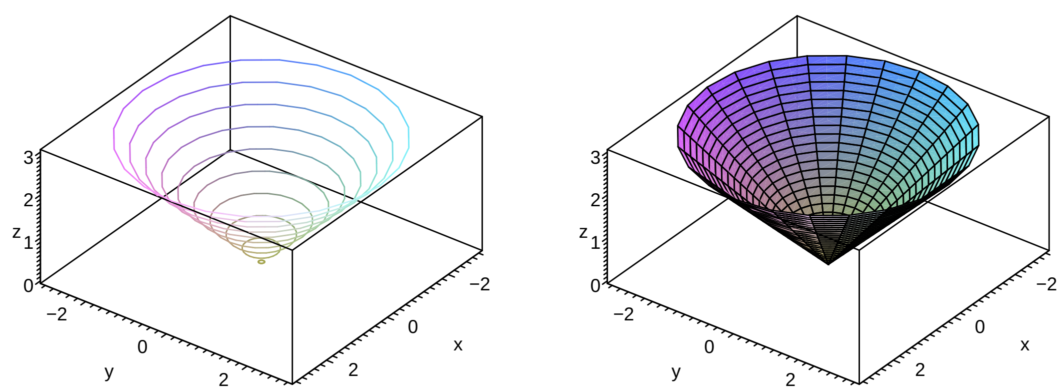

Figure 15.4 Tracing a surface.

EXAMPLE 15.14 Let $\mathbf{r} = \langle v \cos u, v \sin u, u\rangle$. If v is constant, the resulting curve is a helix (as in figure 12.1). If u is constant, the resulting curve is a straight line at height u in the direction u radians from the positive x axis. Note in figure 15.5 how the helixes and the lines both paint the same surface in a different way. □

This technique allows us to represent many more surfaces than previously.

EXAMPLE 15.15 The curve given by

$$\mathbf{r} = \langle (2 + \cos(3u/2)) \cos u, (2 + \cos(3u/2)) \sin u, \sin(3u/2)\rangle$$

is called a trefoil knot. Recall that from the vector equation of the curve we can compute the unit tangent $\mathbf{T}$, the unit normal $\mathbf{N}$, and the binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$; you may

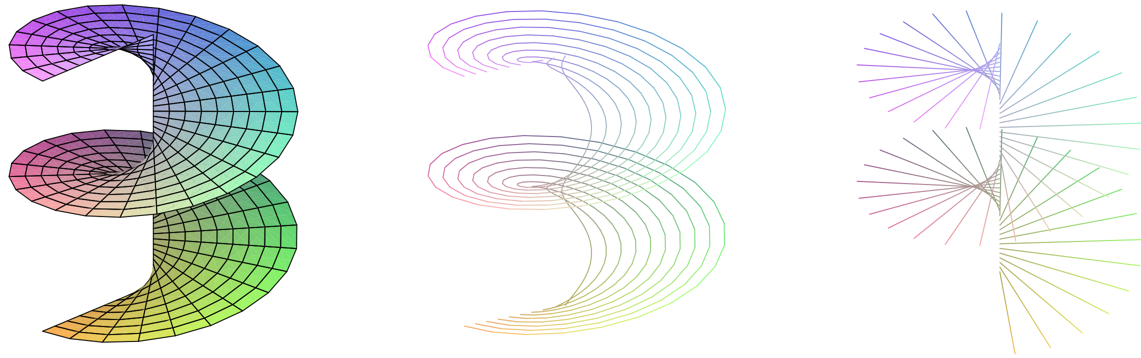

Figure 15.5 Tracing a surface. (JA)

want to review section 12.3. The binormal is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$; one way to interpret this is that $\mathbf{N}$ and $\mathbf{B}$ define a plane perpendicular to $\mathbf{T}$, that is, perpendicular to the curve; since $\mathbf{N}$ and $\mathbf{B}$ are perpendicular to each other, they can function just as $\mathbf{i}$ and $\mathbf{j}$ do for the x-y plane. So, for example, $\mathbf{c}(v) = \mathbf{N}\cos v + \mathbf{B}\sin v$ is a vector equation for a unit circle in a plane perpendicular to the curve described by $\mathbf{r}$, except that the usual interpretation of $\mathbf{c}$ would put its center at the origin. We can fix that simply by adding $\mathbf{c}$ to the original $\mathbf{r}$: let $\mathbf{f} = \mathbf{r}(u) + \mathbf{c}(v)$. For a fixed u this draws a circle around the point $\mathbf{r}(u)$; as u varies we get a sequence of such circles around the curve $\mathbf{r}$, that is, a tube of radius 1 with $\mathbf{r}$ at its center. We can easily change the radius; for example $\mathbf{r}(u) + a\mathbf{c}(v)$ gives the tube radius a; we can make the radius vary as we move along the curve with $\mathbf{r}(u) + g(u)\mathbf{c}(v)$, where $g(u)$ is a function of u. As shown in figure 15.6, it is hard to see that the plain knot is knotted; the tube makes the structure apparent. Of course, there is nothing special about the trefoil knot in this example; we can put a tube around (almost) any curve in the same way. □

We have previously examined surfaces given in the form $f(x, y)$. It is sometimes useful to represent such surfaces in the more general vector form, which is quite easy: $\mathbf{r}(u, v) = \langle u, v, f(u, v)\rangle$. The names of the variables are not important of course; instead of disguising x and y, we could simply write $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$.

We have also previously dealt with surfaces that are not functions of x and y; many of these are easy to represent in vector form. One common type of surface that cannot be represented as $z = f(x, y)$ is a surface given by an equation involving only x and y. For example, $x + y = 1$ and $y = x^2$ are "vertical" surfaces. For every point (x, y) in the plane that satisfies the equation, the point (x, y, z) is on the surface, for every value of z. Thus,

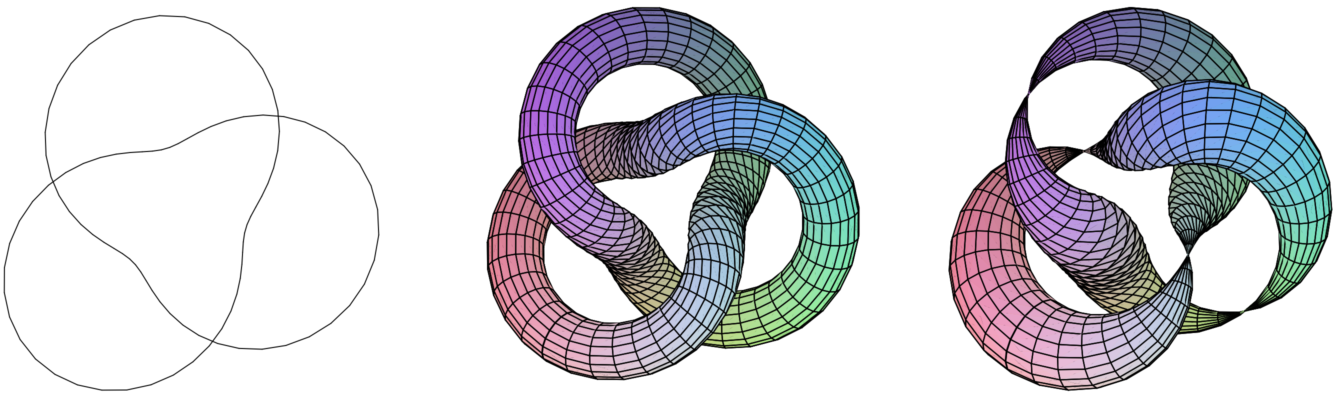

Figure 15.6 Tubes around a trefoil knot, with radius 1/2 and $3\cos(u)/4$. (JA)

a corresponding vector form for the surface is something like $\langle f(u), g(u), v\rangle$; for example, $x+y=1$ becomes $\langle u, 1-u, v\rangle$ and $y=x^2$ becomes $\langle u, u^2, v\rangle$.

Yet another sort of example is the sphere, say $x^2+y^2+z^2=1$. This cannot be written in the form $z=f(x,y)$, but it is easy to write in vector form; indeed this particular surface is much like the cone, since it has circular cross-sections, or we can think of it as a tube around a portion of the z-axis, with a radius that varies depending on where along the axis we are. One vector expression for the sphere is $\langle\sqrt{1-v^2}\cos u, \sqrt{1-v^2}\sin u, v\rangle$—this emphasizes the tube structure, as it is naturally viewed as drawing a circle of radius $\sqrt{1-v^2}$ around the z-axis at height v. We could also take a cue from spherical coordinates, and write $\langle\sin u\cos v, \sin u\sin v, \cos u\rangle$, where in effect u and v are ϕ and θ in disguise.

It is quite simple in Maple to plot any surface for which you have a vector representation. Using different vector functions sometimes gives different looking plots, because Maple in effect draws the surface by holding one variable constant and then the other. For example, you might have noticed in figure 15.5 that the curves in the two right-hand graphs are superimposed on the left-hand graph; the graph of the surface is just the combination of the two sets of curves, with the spaces filled in with color.

Here's a simple but striking example: the plane $x+y+z=1$ can be represented quite naturally as $\langle u, v, 1-u-v\rangle$. But we could also think of painting the same plane by choosing a particular point on the plane, say $(1,0,0)$, and then drawing circles or ellipses (or any of a number of other curves) as if that point were the origin in the plane. For example, $\langle 1-v\cos u-v\sin u, v\sin u, v\cos u\rangle$ is one such vector function. Note that while it may not be obvious where this came from, it is quite easy to see that the sum of the x,

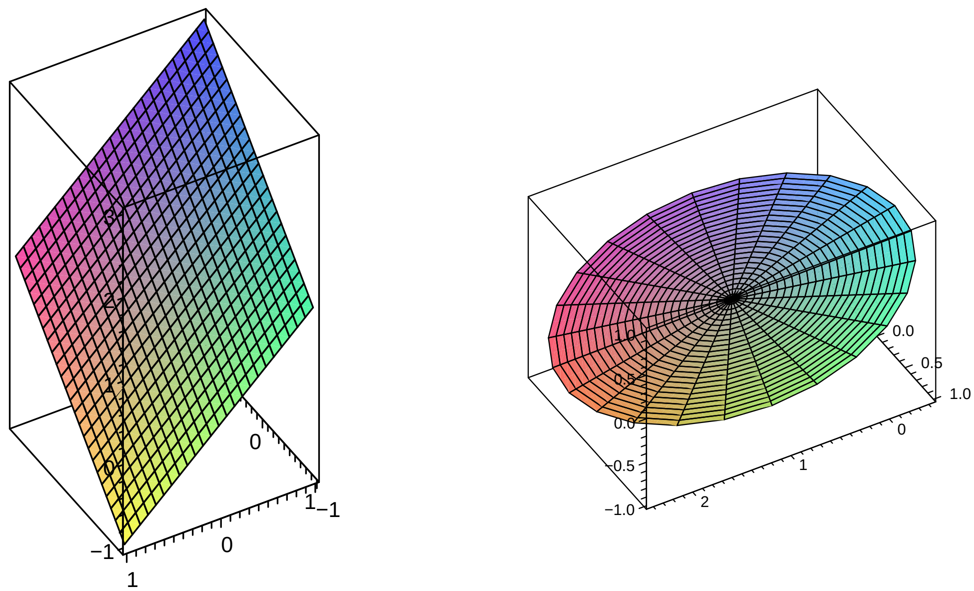

Figure 15.7 Two representations of the same plane. (JA)

y, and z components of the vector is always 1. Maple's renderings of the plane using these two functions are shown in figure 15.7.

Suppose we know that a plane contains a particular point (x_0, y_0, z_0) and that two vectors $\mathbf{u} = \langle u_0, u_1, u_2\rangle$ and $\mathbf{v} = \langle v_0, v_1, v_2\rangle$ are parallel to the plane but not to each other. We know how to get an equation for the plane in the form $ax + by + cz = d$, by first computing $\mathbf{u} \times \mathbf{v}$. It's even easier to get a vector equation:

$$\mathbf{r}(u, v) = \langle x_0, y_0, z_0\rangle + u\mathbf{u} + v\mathbf{v}.$$

The first vector gets to the point (x_0, y_0, z_0) and then by varying u and v, $u\mathbf{u} + v\mathbf{v}$ gets to every point in the plane.

Returning to $x + y + z = 1$, the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are all on the plane. By subtracting coordinates we see that $\langle -1, 0, 1\rangle$ and $\langle -1, 1, 0\rangle$ are parallel to the plane, so a third vector form for this plane is

$$\langle 1, 0, 0\rangle + u\langle -1, 0, 1\rangle + v\langle -1, 1, 0\rangle = \langle 1 - u - v, v, u\rangle.$$

This is clearly quite similar to the first form we found.

We have already seen (section 14.4) how to find the area of a surface when it is defined in the form $f(x, y)$. Finding the area when the surface is given as a vector function is very similar. Looking at the plots of surfaces we have just seen, it is evident that the two sets of curves that fill out the surface divide it into a grid, and that the spaces in the grid are approximately parallelograms. As before this is the key: we can write down the area of a typical little parallelogram and add them all up with an integral.

Suppose we want to approximate the area of the surface $\mathbf{r}(u,v)$ near $\mathbf{r}(u_0,v_0)$. The functions $\mathbf{r}(u,v_0)$ and $\mathbf{r}(u_0,v)$ define two curves that intersect at $\mathbf{r}(u_0,v_0)$. The derivatives of $\mathbf{r}$ give us vectors tangent to these two curves: $\mathbf{r}_u(u_0,v_0)$ and $\mathbf{r}_v(u_0,v_0)$, and then $\mathbf{r}_u(u_0,v_0)\,du$ and $\mathbf{r}_v(u_0,v_0)\,dv$ are two small tangent vectors, whose lengths can be used as the lengths of the sides of an approximating parallelogram. Finally, the area of this parallelogram is $|\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv$ and so the total surface area is

$$\int_a^b\int_c^d |\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv.$$

EXAMPLE 15.16 We find the area of the surface $\langle v\cos u, v\sin u, u\rangle$ for $0\le u\le\pi$ and $0\le v\le 1$; this is a portion of the helical surface in figure 15.5. We compute $\mathbf{r}_u=\langle -v\sin u, v\cos u, 1\rangle$ and $\mathbf{r}_v=\langle\cos u,\sin u,0\rangle$. The cross product of these two vectors is $\langle\sin u,-\cos u,v\rangle$ with length $\sqrt{1+v^2}$, and the surface area is

$$\int_0^\pi\int_0^1\sqrt{1+v^2}\,dv\,du=\frac{\pi\sqrt{2}}{2}+\frac{\pi\ln(\sqrt{2}+1)}{2}.$$

□

Exercises

1. Find the area of the portion of $x+2y+4z=10$ in the first octant. ⇒

2. Find the area of the portion of $2x+4y+z=0$ inside $x^2+y^2=1$. ⇒

3. Find the area of $z=x^2+y^2$ that lies below $z=1$. ⇒

4. Find the area of $z=\sqrt{x^2+y^2}$ that lies below $z=2$. ⇒

5. Find the area of the portion of $x^2+y^2+z^2=a^2$ that lies in the first octant. ⇒

6. Find the area of the portion of $x^2+y^2+z^2=a^2$ that lies above $x^2+y^2\le b^2$. ⇒

7. Find the area of $z=x^2-y^2$ that lies inside $x^2+y^2=a^2$. ⇒

8. Find the area of $z=xy$ that lies inside $x^2+y^2=a^2$. ⇒

9. Find the area of $x^2+y^2+z^2=a^2$ that lies above the interior of the circle given in polar coordinates by $r=a\cos\theta$. ⇒

10. Find the area of the cone $z=k\sqrt{x^2+y^2}$ that lies above the interior of the circle given in polar coordinates by $r=a\cos\theta$. ⇒

11. Find the area of the plane $z=ax+by+c$ that lies over a region D with area A. ⇒

12. Find the area of the cone $z = k\sqrt{x^2 + y^2}$ that lies over a region D with area A. ⇒

13. Find the area of the cylinder $x^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = a^2$. ⇒

14. The surface $f(x, y)$ can be represented with the vector function $\langle x, y, f(x, y)\rangle$. Set up the surface area integral using this vector function and compare to the integral of section 14.4.

15.7 Surface Integrals

In the integral for surface area,

$$\int_a^b \int_c^d |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv,$$

the integrand $|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$ is the area of a tiny parallelogram, that is, a very small surface area, so it is reasonable to abbreviate it dS; then a shortened version of the integral is

$$\iint_D 1 \cdot dS.$$

We have already seen that if D is a region in the plane, the area of D may be computed with

$$\iint_D 1 \cdot dA,$$

so this is really quite familiar, but the dS hides a little more detail than does dA.

Just as we can integrate functions $f(x, y)$ over regions in the plane, using

$$\iint_D f(x, y)\,dA,$$

so we can compute integrals over surfaces in space, using

$$\iint_D f(x, y, z)\,dS.$$

In practice this means that we have a vector function $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ for the surface, and the integral we compute is

$$\int_a^b \int_c^d f(x(u, v), y(u, v), z(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv.$$

That is, we express everything in terms of u and v, and then we can do an ordinary double integral.

EXAMPLE 15.17 Suppose a thin object occupies the upper hemisphere of $x^2 + y^2 + z^2 = 1$ and has density $\sigma(x, y, z) = z$. Find the mass and center of mass of the object. (Note that the object is just a thin shell; it does not occupy the interior of the hemisphere.)

We write the hemisphere as $\mathbf{r}(\phi, \theta) = \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$, $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$. So $\mathbf{r}_\theta = \langle -\sin\theta \sin\phi, \cos\theta \sin\phi, 0 \rangle$ and $\mathbf{r}_\phi = \langle \cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi \rangle$. Then

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \langle -\cos\theta \sin^2\phi, -\sin\theta \sin^2\phi, -\cos\phi \sin\phi \rangle$$

and

$$|\mathbf{r}_\theta \times \mathbf{r}_\phi| = |\sin\phi| = \sin\phi,$$

since we are interested only in $0 \le \phi \le \pi/2$. Finally, the density is $z = \cos\phi$ and the integral for mass is

$$\int_0^{2\pi} \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi \, d\theta = \pi.$$

By symmetry, the center of mass is clearly on the z-axis, so we only need to find the z-coordinate of the center of mass. The moment around the x-y plane is

$$\int_0^{2\pi} \int_0^{\pi/2} z \cos\phi \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi \, d\theta = \frac{2\pi}{3},$$

so the center of mass is at $(0, 0, 2/3)$. □

Now suppose that $\mathbf{F}$ is a vector field; imagine that it represents the velocity of some fluid at each point in space. We would like to measure how much fluid is passing through a surface D, the **flux** across D. As usual, we imagine computing the flux across a very small section of the surface, with area dS, and then adding up all such small fluxes over D with an integral. Suppose that vector $\mathbf{N}$ is a unit normal to the surface at a point; $\mathbf{F} \cdot \mathbf{N}$ is the scalar projection of $\mathbf{F}$ onto the direction of $\mathbf{N}$, so it measures how fast the fluid is moving across the surface. In one unit of time the fluid moving across the surface will fill a volume of $\mathbf{F} \cdot \mathbf{N} \, dS$, which is therefore the rate at which the fluid is moving across a small patch of the surface. Thus, the total flux across D is

$$\iint_D \mathbf{F} \cdot \mathbf{N} \, dS = \iint_D \mathbf{F} \cdot d\mathbf{S},$$

defining $d\mathbf{S} = \mathbf{N}\,dS$. As usual, certain conditions must be met for this to work out; chief among them is the nature of the surface. As we integrate over the surface, we must choose the normal vectors $\mathbf{N}$ in such a way that they point "the same way" through the surface. For example, if the surface is roughly horizontal in orientation, we might want to measure the flux in the "upwards" direction, or if the surface is closed, like a sphere, we might want to measure the flux "outwards" across the surface. In the first case we would choose $\mathbf{N}$ to have positive z component, in the second we would make sure that $\mathbf{N}$ points away from the origin. Unfortunately, there are surfaces that are not **orientable**: they have only one side, so that it is not possible to choose the normal vectors to point in the "same way" through the surface. The most famous such surface is the Möbius strip shown in figure 15.8. It is quite easy to make such a strip with a piece of paper and some tape. If you have never done this, it is quite instructive; in particular, you should draw a line down the center of the strip until you return to your starting point. No matter how unit normal vectors are assigned to the points of the Möbius strip, there will be normal vectors very close to each other pointing in opposite directions.

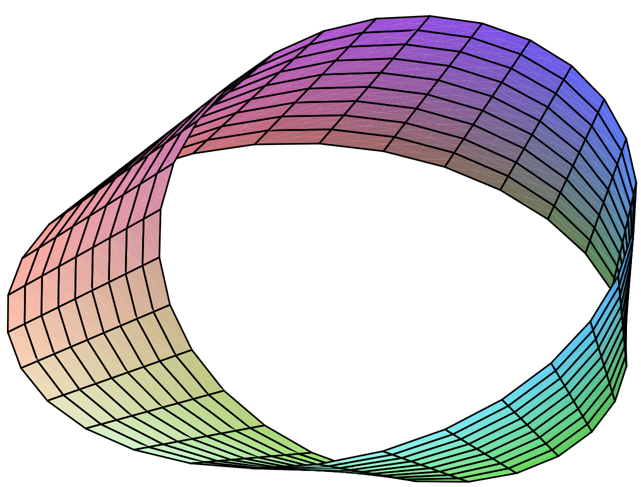

Figure 15.8 A Möbius strip. (JA)

Assuming that the quantities involved are well behaved, however, the flux of the vector field across the surface $\mathbf{r}(u,v)$ is

$$\iint\limits_D \mathbf{F}\cdot\mathbf{N}\,dS = \iint\limits_D \mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}|\mathbf{r}_u\times\mathbf{r}_v|\,dA = \iint\limits_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.$$

In practice, we may have to use $\mathbf{r}_v\times\mathbf{r}_u$ or even something a bit more complicated to make sure that the normal vector points in the desired direction.

EXAMPLE 15.18 Compute the flux of $\mathbf{F}=\langle x,y,z^4\rangle$ across the cone $z=\sqrt{x^2+y^2}$, $0\le z\le 1$, in the downward direction.

We write the cone as a vector function: $\mathbf{r}=\langle v\cos u, v\sin u, v\rangle$, $0\le u\le 2\pi$ and $0\le v\le 1$. Then $\mathbf{r}_u=\langle -v\sin u, v\cos u, 0\rangle$ and $\mathbf{r}_v=\langle\cos u,\sin u,1\rangle$ and $\mathbf{r}_u\times\mathbf{r}_v=\langle v\cos u, v\sin u, -v\rangle$. The third coordinate $-v$ is negative, which is exactly what we desire, that is, the normal vector points down through the surface. Then

$$\begin{aligned}\int_0^{2\pi}\int_0^1 \langle x,y,z^4\rangle\cdot\langle v\cos u, v\sin u,-v\rangle\,dv\,du &= \int_0^{2\pi}\int_0^1 xv\cos u+yv\sin u - z^4v\,dv\,du\\ &= \int_0^{2\pi}\int_0^1 v^2\cos^2 u+v^2\sin^2 u - v^5\,dv\,du\\ &= \int_0^{2\pi}\int_0^1 v^2-v^5\,dv\,du = \frac{\pi}{3}.\end{aligned}$$

□

Exercises

1. Find the center of mass of an object that occupies the upper hemisphere of $x^2+y^2+z^2=1$ and has density x^2+y^2. ⇒
2. Find the center of mass of an object that occupies the surface $z=xy$, $0\le x,y\le 1$, and has density $\sqrt{1+x^2+y^2}$. ⇒
3. Find the centroid of the surface of a right circular cone of height h and base radius r, not including the base. ⇒
4. Evaluate $\displaystyle\iint\limits_D \langle 2,-3,4\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=x^2+y^2$, $-1\le x\le 1$, $-1\le y\le 1$, oriented up. ⇒
5. Evaluate $\displaystyle\iint\limits_D \langle x,y,3\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=3x-5y$, $1\le x\le 2$, $0\le y\le 2$, oriented up. ⇒

6. Evaluate $\displaystyle\iint_D \langle x, y, -2\rangle \cdot \mathbf{N}\,dS$, where D is given by $z = 1 - x^2 - y^2$, $x^2 + y^2 \le 1$, oriented up. ⇒

7. Evaluate $\displaystyle\iint_D \langle xy, yz, zx\rangle \cdot \mathbf{N}\,dS$, where D is given by $z = x + y^2 + 2$, $0 \le x \le 1$, $x \le y \le 1$, oriented up. ⇒

8. Evaluate $\displaystyle\iint_D \langle e^x, e^y, z\rangle \cdot \mathbf{N}\,dS$, where D is given by $z = xy$, $0 \le x \le 1$, $-x \le y \le x$, oriented up. ⇒

9. Evaluate $\displaystyle\iint_D \langle xz, yz, z\rangle \cdot \mathbf{N}\,dS$, where D is given by $z = a^2 - x^2 - y^2$, $x^2 + y^2 \le b^2$, oriented up. ⇒

15.8 Stokes's Theorem

Recall that one version of Green's Theorem (see equation 15.2) is

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA.$$

Here D is a region in the x-y plane and $\mathbf{k}$ is a unit normal to D at every point. If D is instead an orientable surface in space, there is an obvious way to alter this equation, and it turns out still to be true:

THEOREM 15.19 Stokes's Theorem Provided that the quantities involved are sufficiently nice, and in particular if D is orientable,

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\,dS,$$

if ∂D is oriented counter-clockwise relative to $\mathbf{N}$.

Note how little has changed: $\mathbf{k}$ becomes $\mathbf{N}$, a unit normal to the surface, and dA becomes dS, since this is now a general surface integral. The phrase "counter-clockwise relative to $\mathbf{N}$" means that if we take the direction of $\mathbf{N}$ to be "up", then we go around the boundary counter-clockwise when viewed from "above".

EXAMPLE 15.20 Let $\mathbf{F} = \langle e^{xy}\cos z, x^2 z, xy\rangle$ and the surface D be $x = \sqrt{1 - y^2 - z^2}$, oriented in the positive x direction. It quickly becomes apparent that the surface integral in Stokes's Theorem is intractable, so we try the line integral. The boundary of D is the unit circle in the y-z plane, $\mathbf{r} = \langle 0, \cos u, \sin u\rangle$, $0 \le u \le 2\pi$. The integral is

$$\int_0^{2\pi} \langle e^{xy}\cos z, x^2 z, xy\rangle \cdot \langle 0, -\sin u, \cos u\rangle\, du = \int_0^{2\pi} 0\, du = 0,$$

because $x = 0$. □

An interesting consequence of Stokes's Theorem is that if D and E are two orientable surfaces with the same boundary, then

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial E} \mathbf{F} \cdot d\mathbf{r} = \iint_E (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS.$$

Sometimes both of the integrals

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS \qquad \text{and} \qquad \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

are difficult, but you may be able to find a second surface E so that

$$\iint_E (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

has the same value but is easier to compute.

EXAMPLE 15.21 In the previous example, the line integral was easy to compute. But we might also notice that another surface E with the same boundary is the flat disk $y^2 + z^2 \le 1$. The unit normal $\mathbf{N}$ for this surface is simply $\mathbf{i} = \langle 1, 0, 0\rangle$. We compute the curl:

$$\nabla \times \mathbf{F} = \langle x - x^2, -e^{xy}\sin z - y, 2xz - xe^{xy}\cos z\rangle.$$

Since $x = 0$ everywhere on the surface,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{N} = \langle 0, -e^{xy}\sin z - y, 2xz - xe^{xy}\cos z\rangle \cdot \langle 1, 0, 0\rangle = 0,$$

so the surface integral is

$$\iint_E 0\, dS = 0,$$

as before. In this case, of course, it is still somewhat easier to compute the line integral, avoiding $\nabla \times \mathbf{F}$ entirely. □

EXAMPLE 15.22 Let $\mathbf{F} = \langle -y^2, x, z^2 \rangle$, and let the curve C be the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$, oriented counter-clockwise when viewed from above. We compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.

First we do it directly: a vector function for C is $\mathbf{r} = \langle \cos u, \sin u, 2 - \sin u \rangle$, so $\mathbf{r}' = \langle -\sin u, \cos u, -\cos u \rangle$, and the integral is then

$$\int_0^{2\pi} y^2 \sin u + x \cos u - z^2 \cos u \, du = \int_0^{2\pi} \sin^3 u + \cos^2 u - (2 - \sin u)^2 \cos u \, du = \pi.$$

To use Stokes's Theorem, we pick a surface with C as the boundary; the simplest such surface is that portion of the plane $y + z = 2$ inside the cylinder. This has vector equation $\mathbf{r} = \langle v \cos u, v \sin u, 2 - v \sin u \rangle$. We compute $\mathbf{r}_u = \langle -v \sin u, v \cos u, -v \cos u \rangle$, $\mathbf{r}_v = \langle \cos u, \sin u, -\sin u \rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, -v, -v \rangle$. To match the orientation of C we need to use the normal $\langle 0, v, v \rangle$. The curl of $\mathbf{F}$ is $\langle 0, 0, 1 + 2y \rangle = \langle 0, 0, 1 + 2v \sin u \rangle$, and the surface integral from Stokes's Theorem is

$$\int_0^{2\pi} \int_0^1 (1 + 2v \sin u) v \, dv \, du = \pi.$$

In this case the surface integral was more work to set up, but the resulting integral is somewhat easier. □

Proof of Stokes's Theorem.

We can prove here a special case of Stokes's Theorem, which perhaps not too surprisingly uses Green's Theorem.

Suppose the surface D of interest can be expressed in the form $z = g(x, y)$, and let $\mathbf{F} = \langle P, Q, R \rangle$. Using the vector function $\mathbf{r} = \langle x, y, g(x, y) \rangle$ for the surface we get the surface integral

$$\begin{aligned}\iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_E \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \cdot \langle -g_x, -g_y, 1 \rangle \, dA \\ &= \iint_E -R_y g_x + Q_z g_x - P_z g_y + R_x g_y + Q_x - P_y \, dA.\end{aligned}$$

Here E is the region in the x-y plane directly below the surface D.

For the line integral, we need a vector function for ∂D. If $\langle x(t), y(t)\rangle$ is a vector function for ∂E then we may use $\mathbf{r}(t) = \langle x(t), y(t), g(x(t), y(t))\rangle$ to represent ∂D. Then

$$\int_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \int_a^b P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\,dt = \int_a^b P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\,dt.$$

using the chain rule for dz/dt. Now we continue to manipulate this:

$$\begin{aligned}
\int_a^b P\frac{dx}{dt} + Q\frac{dy}{dt} &+ R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\,dt \\
&= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right]\,dt \\
&= \int_{\partial E} \left(P + R\frac{\partial z}{\partial x}\right)\,dx + \left(Q + R\frac{\partial z}{\partial y}\right)\,dy,
\end{aligned}$$

which now looks just like the line integral of Green's Theorem, except that the functions P and Q of Green's Theorem have been replaced by the more complicated $P + R(\partial z/\partial x)$ and $Q + R(\partial z/\partial y)$. We can apply Green's Theorem to get

$$\int_{\partial E} \left(P + R\frac{\partial z}{\partial x}\right)\,dx + \left(Q + R\frac{\partial z}{\partial y}\right)\,dy = \iint\limits_E \frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\,dA.$$

Now we can use the chain rule again to evaluate the derivatives inside this integral, and it becomes

$$\begin{aligned}
&\iint\limits_E Q_x + Q_z g_x + R_x g_y + R_z g_x g_y + R g_{yx} - (P_y + P_z g_y + R_y g_x + R_z g_y g_x + R g_{xy})\,dA \\
&\qquad = \iint\limits_E Q_x + Q_z g_x + R_x g_y - P_y - P_z g_y - R_y g_x\,dA,
\end{aligned}$$

which is the same as the expression we obtained for the surface integral. ■

Exercises

1. Let $\mathbf{F} = \langle z, x, y\rangle$. The plane $z = 2x + 2y - 1$ and the paraboloid $z = x^2 + y^2$ intersect in a closed curve. Stokes's Theorem implies that

$$\iint_{D_1} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{D_2} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS,$$

where the line integral is computed over the intersection C of the plane and the paraboloid, and the two surface integrals are computed over the portions of the two surfaces that have boundary C (provided, of course, that the orientations all match). Compute all three integrals. ⇒

2. Let D be the portion of $z = 1 - x^2 - y^2$ above the x-y plane, oriented up, and let $\mathbf{F} = \langle xy^2, -x^2y, xyz\rangle$. Compute $\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$. ⇒

3. Let D be the portion of $z = 2x + 5y$ inside $x^2 + y^2 = 1$, oriented up, and let $\mathbf{F} = \langle y, z, -x\rangle$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$. ⇒

4. Let D be the portion of $z = px + qy + r$ over a region in the x-y plane that has area A, oriented up, and let $\mathbf{F} = \langle ax + by + cz, ax + by + cz, ax + by + cz\rangle$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$. ⇒

5. Let D be any surface and let $\mathbf{F} = \langle P(x), Q(y), R(z)\rangle$ (P depends only on x, Q only on y, and R only on z). Show that $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = 0$.

6. Show that $\int_C f\nabla g + g\nabla f \cdot d\mathbf{r} = 0$, where $\mathbf{r}$ describes a closed curve C to which Stokes's Theorem applies.

15.9 The Divergence Theorem

The third version of Green's Theorem (equation 15.3) we saw was:

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA.$$

With minor changes this turns into another equation, the Divergence Theorem:

THEOREM 15.23 Divergence Theorem Under suitable conditions, if E is a region of three dimensional space and D is its boundary surface, oriented outward, then

$$\iint_D \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_E \nabla \cdot \mathbf{F}\, dV.$$

Proof of the Divergence Theorem.

Again this theorem is too difficult to prove here, but a special case is easier. In the proof of a special case of Green's Theorem, we needed to know that we could describe the region of integration in both possible orders, so that we could set up one double integral using $dx\,dy$ and another using $dy\,dx$. Similarly here, we need to be able to describe the three-dimensional region E in different ways.

We start by rewriting the triple integral:

$$\iiint_E \nabla\cdot\mathbf{F}\,dV = \iiint_E (P_x + Q_y + R_z)\,dV = \iiint_E P_x\,dV + \iiint_E Q_y\,dV + \iiint_E R_z\,dV.$$

The double integral may be rewritten:

$$\iint_D \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D (P\mathbf{i}+Q\mathbf{j}+R\mathbf{k})\cdot\mathbf{N}\,dS = \iint_D P\mathbf{i}\cdot\mathbf{N}\,dS + \iint_D Q\mathbf{j}\cdot\mathbf{N}\,dS + \iint_D R\mathbf{k}\cdot\mathbf{N}\,dS.$$

To prove that these give the same value it is sufficient to prove that

$$\begin{aligned} \iint_D P\mathbf{i}\cdot\mathbf{N}\,dS &= \iiint_E P_x\,dV,\\ \iint_D Q\mathbf{j}\cdot\mathbf{N}\,dS &= \iiint_E Q_y\,dV, \text{ and}\\ \iint_D R\mathbf{k}\cdot\mathbf{N}\,dS &= \iiint_E R_z\,dV. \end{aligned} \tag{15.4}$$

Not surprisingly, these are all pretty much the same; we'll do the first one.

We set the triple integral up with dx innermost:

$$\iiint_E P_x\,dV = \iint_B \int_{g_1(y,z)}^{g_2(y,z)} P_x\,dx\,dA = \iint_B P(g_2(y,z),y,z) - P(g_1(y,z),y,z)\,dA,$$

where B is the region in the y-z plane over which we integrate. The boundary surface of E consists of a "top" $x = g_2(y,z)$, a "bottom" $x = g_1(y,z)$, and a "wrap-around side" that is vertical to the y-z plane. To integrate over the entire boundary surface, we can integrate over each of these (top, bottom, side) and add the results. Over the side surface, the vector $\mathbf{N}$ is perpendicular to the vector $\mathbf{i}$, so

$$\iint_{\text{side}} P\mathbf{i}\cdot\mathbf{N}\,dS = \iint_{\text{side}} 0\,dS = 0.$$

Thus, we are left with just the surface integral over the top plus the surface integral over the bottom. For the top, we use the vector function $\mathbf{r} = \langle g_2(y,z), y, z\rangle$ which gives $\mathbf{r}_y \times \mathbf{r}_z = \langle 1, -g_{2y}, -g_{2z}\rangle$; the dot product of this with $\mathbf{i} = \langle 1, 0, 0\rangle$ is 1. Then

$$\iint_{\text{top}} P\mathbf{i}\cdot\mathbf{N}\,dS = \iint_B P(g_2(y,z), y, z)\,dA.$$

In almost identical fashion we get

$$\iint_{\text{bottom}} P\mathbf{i}\cdot\mathbf{N}\,dS = -\iint_B P(g_1(y,z), y, z)\,dA,$$

where the negative sign is needed to make $\mathbf{N}$ point in the negative x direction. Now

$$\iint_D P\mathbf{i}\cdot\mathbf{N}\,dS = \iint_B P(g_2(y,z), y, z)\,dA - \iint_B P(g_1(y,z), y, z)\,dA,$$

which is the same as the value of the triple integral above. ∎

EXAMPLE 15.24 Let $\mathbf{F} = \langle 2x, 3y, z^2\rangle$, and consider the three-dimensional volume inside the cube with faces parallel to the principal planes and opposite corners at $(0,0,0)$ and $(1,1,1)$. We compute the two integrals of the divergence theorem.

The triple integral is the easier of the two:

$$\int_0^1\int_0^1\int_0^1 2 + 3 + 2z\,dx\,dy\,dz = 6.$$

The surface integral must be separated into six parts, one for each face of the cube. One face is $z = 0$ or $\mathbf{r} = \langle u, v, 0\rangle$, $0 \le u, v \le 1$. Then $\mathbf{r}_u = \langle 1, 0, 0\rangle$, $\mathbf{r}_v = \langle 0, 1, 0\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, 1\rangle$. We need this to be oriented downward (out of the cube), so we use $\langle 0, 0, -1\rangle$ and the corresponding integral is

$$\int_0^1\int_0^1 -z^2\,du\,dv = \int_0^1\int_0^1 0\,du\,dv = 0.$$

Another face is $y = 1$ or $\mathbf{r} = \langle u, 1, v\rangle$. Then $\mathbf{r}_u = \langle 1, 0, 0\rangle$, $\mathbf{r}_v = \langle 0, 0, 1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, -1, 0\rangle$. We need a normal in the positive y direction, so we convert this to $\langle 0, 1, 0\rangle$, and the corresponding integral is

$$\int_0^1\int_0^1 3y\,du\,dv = \int_0^1\int_0^1 3\,du\,dv = 3.$$

The remaining four integrals have values 0, 0, 2, and 1, and the sum of these is 6, in agreement with the triple integral. □

EXAMPLE 15.25 Let $\mathbf{F} = \langle x^3, y^3, z^2 \rangle$, and consider the cylindrical volume $x^2+y^2 \leq 9$, $0 \leq z \leq 2$. The triple integral (using cylindrical coordinates) is

$$\int_0^{2\pi} \int_0^3 \int_0^2 (3r^2 + 2z)r \, dz \, dr \, d\theta = 279\pi.$$

For the surface we need three integrals. The top of the cylinder can be represented by $\mathbf{r} = \langle v \cos u, v \sin u, 2 \rangle$; $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, -v \rangle$, which points down into the cylinder, so we convert it to $\langle 0, 0, v \rangle$. Then

$$\int_0^{2\pi} \int_0^3 \langle v^3 \cos^3 u, v^3 \sin^3 u, 4 \rangle \cdot \langle 0, 0, v \rangle \, dv \, du = \int_0^{2\pi} \int_0^3 4v \, dv \, du = 36\pi.$$

The bottom is $\mathbf{r} = \langle v \cos u, v \sin u, 0 \rangle$; $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, -v \rangle$ and

$$\int_0^{2\pi} \int_0^3 \langle v^3 \cos^3 u, v^3 \sin^3 u, 0 \rangle \cdot \langle 0, 0, -v \rangle \, dv \, du = \int_0^{2\pi} \int_0^3 0 \, dv \, du = 0.$$

The side of the cylinder is $\mathbf{r} = \langle 3 \cos u, 3 \sin u, v \rangle$; $\mathbf{r}_u \times \mathbf{r}_v = \langle 3 \cos u, 3 \sin u, 0 \rangle$ which does point outward, so

$$\int_0^{2\pi} \int_0^2 \langle 27 \cos^3 u, 27 \sin^3 u, v^2 \rangle \cdot \langle 3 \cos u, 3 \sin u, 0 \rangle \, dv \, du$$
$$= \int_0^{2\pi} \int_0^2 81 \cos^4 u + 81 \sin^4 u \, dv \, du = 243\pi.$$

The total surface integral is thus $36\pi + 0 + 243\pi = 279\pi$. □

Exercises

1. Let E be the volume described by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, and $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N} \, dS$. ⇒

2. Let E be the volume described by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and $\mathbf{F} = \langle 2xy, 3xy, ze^{x+y} \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N} \, dS$. ⇒

3. Let E be the volume described by $0 \le x \le 1$, $0 \le y \le x$, $0 \le z \le x + y$, and $\mathbf{F} = \langle x, 2y, 3z \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N} \, dS$. ⇒

4. Let E be the volume described by $x^2 + y^2 + z^2 \le 4$, and $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N} \, dS$. ⇒

5. Let E be the hemisphere described by $0 \le z \le \sqrt{1 - x^2 - y^2}$, and $\mathbf{F} = \langle \sqrt{x^2 + y^2 + z^2}, \sqrt{x^2 + y^2 + z^2}, \sqrt{x^2 + y^2 + z^2} \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N} \, dS$. ⇒

6. Let E be the volume described by $x^2 + y^2 \le 1$, $0 \le z \le 4$, and $\mathbf{F} = \langle xy^2, yz, x^2 z \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N} \, dS$. ⇒

7. Let E be the solid cone above the x-y plane and inside $z = 1 - \sqrt{x^2 + y^2}$, and $\mathbf{F} = \langle x \cos^2 z, y \sin^2 z, \sqrt{x^2 + y^2} z \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N} \, dS$. ⇒

8. Prove the other two equations in the display 15.4.

14.2.9. $\sqrt{3}/4+\pi/6$

14.3.1. $\bar{x}=\bar{y}=2/3$

14.3.2. $\bar{x}=4/5,\ \bar{y}=8/15$

14.3.3. $\bar{x}=0,\ \bar{y}=3\pi/16$

14.3.4. $\bar{x}=0,\ \bar{y}=16/(15\pi)$

14.4.1. $\pi a\sqrt{h^2+a^2}$

14.4.2. $\pi a^2\sqrt{m^2+1}$

14.4.3. $\sqrt{3}/2$

14.4.4. $\pi\sqrt{2}$

14.4.5. $\pi\sqrt{2}/8$

14.4.6. $\pi/2-1$

14.5.1. $11/24$

14.5.2. $623/60$

14.5.3. $-3e^2/4+2e-3/4$

14.5.4. $1/20$

14.5.5. $\pi/48$

14.5.6. $11/84$

14.5.7. $151/60$

14.5.9. 32

14.5.10. $64/3$

14.5.11. $\bar{x}=\bar{y}=0,\ \bar{z}=16/15$

14.5.12. $\bar{x}=\bar{y}=0,\ \bar{z}=1/3$

14.6.1. $\pi/12$

14.6.2. $5\pi/4$

14.6.3. 0

14.6.4. $5\pi/4$

14.6.5. $4/5$

14.6.6. $256\pi/15$

14.6.7. $4\pi^2$

14.6.8. $\pi k h^2 a^2/12$

14.6.9. $\pi k h a^3/6$

14.6.10. $\pi^2/4$

14.6.11. $4\pi/5$

14.6.12. $124\pi/5$

14.7.2. 0

14.7.3. $2/3$

14.7.6. $32(\sqrt{2}+\ln(1+\sqrt{2}))/3$

14.7.7. $3\cos(1)-3\cos(4)$

15.2.1. $13\sqrt{11}/4$

15.2.2. 0

15.2.3. $3\sin(4)/2$

15.2.4. 0

15.2.5. $2e^3$

15.2.6. 128

15.2.7. $(9e-3)/2$

15.2.8. $e^{e+1}-e^e-e^{1/e-1}+e^{1/e}+e^4/4-e^{-4}/4$

15.2.9. $1+\sin(1)-\cos(1)$

15.2.10. $3\ln 3-2\ln 2$

15.2.11. $3/20+10\ln(2)/7$

15.2.12. $2\ln 5-2\ln 2+15/32$

15.2.13. 1

15.2.14. 0

15.2.15. $21+\cos(1)-\cos(8)$

15.2.16. $(\ln 29-\ln 2)/2$

15.2.17. $2\ln 2+\pi/4-2$

15.2.18. $1243/3$

15.2.19. $\ln 2+11/3$

15.2.20. $3\cos(1)-\cos(2)-\cos(4)-\cos(8)$

15.2.21. $-10/3$

15.3.1. no f

15.3.2. $x^4/4-y^5/5$

15.3.3. no f

15.3.4. no f

15.3.5. $y\sin x$

15.3.6. no f

15.3.7. xyz

15.3.8. 6

15.3.9. $1/e - \sin 3$

15.3.10. $1/\sqrt{77} - \sqrt{3}$

15.4.1. 1

15.4.2. 0

15.4.3. $1/(2e) - 1/(2e^7) + e/2 - e^7/2$

15.4.4. $1/2$

15.4.5. $-1/6$

15.4.6. $(2\sqrt{3}-10\sqrt{5}+8\sqrt{6})/3-2\sqrt{2}/5+1/5$

15.4.7. $11/2 - \ln(2)$

15.4.8. $2 - \pi/2$

15.4.9. $-17/12$

15.4.10. 0

15.4.11. $-\pi/2$

15.4.12. 12π

15.5.1. -1, 0

15.5.2. 0, $a + b$

15.5.3. $(2b - a)/3$, 0

15.5.4. 0, 1

15.5.5. -2π, 0

15.5.6. 0, 2π

15.6.1. $25\sqrt{21}/4$

15.6.2. $\pi\sqrt{21}$

15.6.3. $\pi(5\sqrt{5} - 1)/6$

15.6.4. $4\pi\sqrt{2}$

15.6.5. $\pi a^2/2$

15.6.6. $2\pi a(a - \sqrt{a^2 - b^2})$

15.6.7. $\pi((1 + 4a^2)^{3/2} - 1)/6$

15.6.8. $2\pi((1 + a^2)^{3/2} - 1)/3$

15.6.9. $\pi a^2 - 2a^2$

15.6.10. $\pi a^2\sqrt{1 + k^2}/4$

15.6.11. $A\sqrt{1 + a^2 + b^2}$

15.6.12. $A\sqrt{k^2 + 1}$

15.6.13. $8a^2$

15.7.1. $(0, 0, 3/8)$

15.7.2. $(11/20, 11/20, 3/10)$

15.7.3. on center axis, $h/3$ above the base

15.7.4. 16

15.7.5. 7

15.7.6. $-\pi$

15.7.7. $-137/120$

15.7.8. $-2/e$

15.7.9. $\pi b^2(-4b^4 - 3b^2 + 6a^2b^2 + 6a^2)/6$

15.8.1. -3π

15.8.2. 0

15.8.3. -4π

15.8.4. $A(p(c - b) + q(a - c) + a - b)$

15.9.1. $a^2bc + ab^2c + abc^2$

15.9.2. $e^2 - 2e + 7/2$

15.9.3. 3

15.9.4. $384\pi/5$

15.9.5. $\pi/3$

15.9.6. 10π

15.9.7. $\pi/2$